A Uniqueness Theorem for Boltzmann Equation from the Coagulation-Fragmentation Dynamics

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Abstract

We prove the uniqueness theorem for general coagulation-fragmentation equation. This theorem completes study of correctness of the problem concerned for many important cases where the existence theorems had been already proved.

Introduction

We examine the general coagulation-fragmentation equation which can be written as

$$\frac{\partial c(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy + \int_0^\infty F(x,y)c(x+y,t)dy$$
$$-c(x,t)\int_0^\infty K(x,y)c(y,t)dy - \frac{1}{2}c(x,t)\int_0^x F(x-y,y)dy, \quad (1)$$

$$c(x,0) = c_0(x) \geq 0. \quad (2)$$

Equation (1) describes the distribution function $c(x,t) \geq 0$ of particles of mass $x \geq 0$ at time $t \geq 0$ whose change in mass governed by the non-negative reaction rates K and F which are called, respectively, the coagulation and fragmentation kernels. The coagulation kernel K models the rate at which particles of size x coalesce with those of size y while the kernel F expresses the rate at which particles of size $(x+y)$ fragment into those of sizes x and y. From a physical point of view it is clear that K and F must be nonnegative and symmetric: $K(x,y) = K(y,x) \geq 0$, $F(x,y) = F(y,x) \geq 0$ for all $0 \leq x, y < \infty$; all functions in (1), (2) are to be nonnegative and a solution $c(x,t)$ has to have the bounded first moment

$$\int_0^\infty xc(x,t)dx < \infty$$

which is equal to the total mass of particles. The first two integrals in (1) describe the growth of the number of particles of size x due to coagulation and fragmentation respectively, while other integrals describe the reverse of these processes. A brief physical interpretation of the integrals appearing on the right hand side of equation (1) can be found, e.g., in [3]. Applications of (1) can be found in many problems including chemistry (e.g. reacting polymers), physics (aggregation of colloidal particles, growth of gas bubbles in solids), astrophysics (formation of stars and planets), meteorology (merging of drops in atmospheric clouds). Many papers are devoted to analyse the existence for the problem concerned (e.g. [4, 6, 8, 9, 11, 14]). Results in uniqueness are much more poor.

The uniqueness theorem for the problem concerned was proved by Melzak [11] for bounded coagulation and fragmentation kernels with the additional condition

$$\int_0^x F(x-y,y)dy \leq \text{const}.$$

Dubovski & Stewart [4] established uniqueness for coagulation kernels with linear growth on infinity:

$$K(x,y) \leq k(1+x+y), \; F \leq \text{const} \tag{3}$$

in a class of functions which have bounded integral with a weight $\exp(\lambda x), \lambda > 0$:

$$\int_0^\infty \exp(\lambda x)c(x,t)dx < \infty, \quad 0 \leq t \leq T, \; 0 < \lambda < \Lambda.$$

In [1, 16] the uniqueness for constant coagulation and fragmentation kernels $K = F =$const was demonstrated. It was shown in [8] that for kernels

$K(x,y) \leq k(1+x^\alpha y^\alpha),\ \alpha < 1,\ F = 0$ the problem concerned can have at most one solution in class of functions, integrable with the weight $\exp(\lambda x^\alpha),\ \lambda > 0$. The next step was done by Stewart [15] who proved uniqueness for the kernels

$$K(x,y) \leq k_1\sqrt{1+x}\sqrt{1+y}, \quad \int_0^x \sqrt{1+y}F(x-y,y)dy \leq k_2\sqrt{1+x} \qquad (4)$$

in the natural class of functions with bounded first moment.

1. Main result

Let the coagulation kernels be symmetric and satisfy the following condition. Suppose that for all $x \geq 0$ there exists $X(x) \geq 1$ such that

$$K(x,y) = a(x)y + b(x,y) \quad \text{if } \ y \geq X(x) \qquad (5)$$

and there exist positive constants $\lambda,\ G$ such that

$$\sup_{0\leq y\leq X(x)} K(x,y) + \sup_{y\geq X(x)} b(x,y) + a(x)X(x) \leq G\exp(\lambda x), \quad x \geq 0. \qquad (6)$$

The functions a and b are to be nonnegative. The class (5), (6) includes many physically reasonable coagulation kernels. Particularly, this class has large intersection with coagulation kernels satisfying (3) or (4). Also, the class (5) includes bounded coagulation kernels considered by Melzak [11], linear kernels [4, 6] and multiplicative ones ($K = (Ax+B)(Ay+B)$) which are considered, e.g., in [5, 7, 9]. In addition, this class includes the following kernels:

$$K(x,y) = \alpha(x,y) + \beta(y)x + \beta(x)y + \gamma(x,y)$$

where

$$\gamma(x,y) = \begin{cases} g_1(x)x + g_2(x)y + g_3(x)xy, & y \geq x \\ g_1(y)y + g_2(y)x + g_3(y)xy, & y \leq x. \end{cases}$$

The functions α, β and $g_i,\ \ i = 1,2,3$ are to be nonnegative and bounded.

We shall consider fragmentation kernels which are nonnegative, symmetric and satisfy for positive constants μ and A the following condition:

$$\int_0^x F(x-y,y)\exp(-\mu y)dy \leq A, \qquad x \geq 0. \qquad (7)$$

This class includes bounded and a lot of other fragmentation kernels (e.g. $F(x,y) = (x+y)^{-1}$).

Let us introduce the class Y of nonnegative continuous functions of $(x,t) \in [0,\infty) \times [0,\infty)$ with the same first moment, i.e. for any $c_1, c_2 \in Y$

$$\int_0^\infty x c_1(x,t)dx = \int_0^\infty x c_2(x,t)dx < \infty, \quad t \geq 0. \tag{8}$$

So, the space $C(R_+^2)$ is decomposed to a lot of classes Y depending on their first moment behaviour.

The aim of this paper is to prove the following uniqueness theorem.

Theorem. The initial value problem (1),(2) with a coagulation kernel from the class (5) and a fragmentation kernel from (7) has at most one nonnegative continuous solution in any class Y.

Remark 1. The condition (8) is very natural. In fact, if we multiply (1) by x and integrate it over $x \in [0,\infty)$ then we get the mass conservation law

$$\frac{d}{dt}\int_0^\infty xc(x,t)dx = 0.$$

By deriving this important equality we assumed that all integrals are bounded. The mass conservation takes place, e.g., for coagulation kernels with linear growth on infinity and bounded fragmentation ones [4]. In this case the equality (8) holds almost always [4].

Remark 2. If the kernel K does not satisfy (3) then the mass conservation law can be infringed. This phenomenon is discussed, e.g., in [5, 7, 9, 10, 12] for the important multiplicative case $K = xy, \quad F = 0$. For this case the behaviour of the total mass (which is expressed by the first moment of solution) is uniquely defined. In this case the condition (8) is also always valid, and we obtain "global" uniqueness, too.

Remark 3. In the well-known example of non-uniqueness [15, 17] with $K \equiv 0, \ F \equiv 2, \ c_0(x) = (\lambda + x)^{-3}, \lambda > 0$ there are two solutions

$$c_1(x,t) = \frac{\exp(\lambda t)}{(\lambda+x)^3}, \quad c_2(x,t) = \exp(-tx)\left(c_0(x) + \int_x^\infty c_0(y)[2t + t^2(y-x)]dy\right).$$

The first solution does not satisfy the mass conservation law, but for the second one the mass conservation holds. Therefore the condition (8) is not valid, and this non-uniqueness meets no contradiction with the assertion of the Theorem.

2. Auxiliary statement

Lemma. Let $v(q,t)$ be a real continuous function having continuous partial derivatives v_q and v_{qq} on $D = \{0 < q_0 \leq q \leq q_1,\ 0 \leq t \leq T\}$. Assume that $\alpha(q), \beta(q,t), \gamma(q,t)$ and $\theta(q,t)$ are real continuous functions on D and their first partial derivatives in q are continuous. Let v, v_{qq}, β, γ be nonnegative and $v_q, \alpha_q, \beta_q, \gamma_q, \theta_q$ be nonpositive functions on D. Suppose also, that the following inequalities hold on D:

$$v(q,t) \leq \alpha(q) + \int_0^t (-\beta(q,s)v_q(q,s) + \gamma(q,s)v(q,s) + \theta(q,s))ds; \quad (9)$$

$$v_q(q,t) \geq \alpha_q(q) + \int_0^t \frac{\partial}{\partial q}\left(-\beta(q,s)v_q(q,s) + \gamma(q,s)v(q,s) + \theta(q,s)\right) ds. \quad (10)$$

Let $c_0 = \sup_{q_0 \leq q \leq q_1} \alpha, c_1 = \sup_D \beta,\ c_2 = \sup_D \gamma,\ c_3 = \sup_D \theta$. Then

$$v(q,t) \leq c_0 \exp(c_2 t) + (c_3/c_2)(\exp(c_2 t) - 1)$$

in any region $R \subset D$:

$$R = \ \left\{(q,t) : 0 \leq t \leq T';\ q_0 + c_1 t \leq q \leq q_1 - \varepsilon + c_1 t,\ 0 < \varepsilon < q_1 - q_0, \right\}, \quad T' = \min\{T, \varepsilon/c_1\}.$$

Proof. Let us denote $w(q,t)$ the right-hand side of the inequality (9). By differentiating in t, q, we obtain from (9),(10) taking nonpositivity of w_q into account:

$$w_t \leq -\beta w_q + \gamma w + \theta \leq -c_1 w_q + \gamma w + \theta.$$

Hence for the derivative along the characteristic $dq/dt = c_1$ we have

$$\frac{dw}{dt} \leq \gamma w + \theta. \quad (11)$$

Let us denote $u(t) = \bar{c}_0 \exp(c_2 t) + (\bar{c}_3/c_2)(\exp(c_2 t) - 1)$ with $\bar{c}_0 > c_0, \bar{c}_3 > c_3$. Obviously, $u(0) > w(q,0)$ for all $q_0 \leq q \leq q_1$. Let $(\hat{q}, \hat{t})$ be the first point on a characteristic straight line, where $w = u$. Then in the point $(\hat{q}, \hat{t})$

$$\frac{d(u-w)}{dt} \leq 0$$

and, consequently,

$$w_t + c_1 w_q \geq u_t. \tag{12}$$

From $u_t = c_2 u + \bar{c}_3$ we see that in the point $(\hat{q}, \hat{t})$ the equality $u_t = c_2 w + \bar{c}_3$ holds. Recalling (12), we obtain the contradiction with (11):

$$\frac{dw}{dt} = w_t + c_1 w_q \geq c_2 w + \bar{c}_3 > c_2 w + c_3 \geq \gamma w + \theta.$$

This proves the Lemma.

3. Proof of the Theorem

We prove the Theorem by contradiction. Suppose that there are two distinct continuous solutions c and d of the initial value problem (1), (2) belonging to the same class Y (i.e. with the same initial data and first moment). Let us denote $u = c - d$. Then we obtain from (1):

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)(c+d)(y,t)dy - \\ -u(x,t)\int_0^\infty K(x,y)c(y,t)dy &- d(x,t)\int_0^\infty K(x,y)u(y,t)dy - \\ -\frac{1}{2}u(x,t)\int_0^x F(x-y,y)dy &+ \int_0^\infty F(x,y)u(x+y,t)dy. \end{aligned} \tag{13}$$

Let us write (13) in the following integral form

$$\begin{aligned} u(x,t) = \int_0^t \exp\Big(-\int_s^t \Big[\int_0^\infty K(x,y)c(y,\tau)dy + \frac{1}{2}\int_0^x F(x-y,y)dy\Big]d\tau\Big) \\ \times\Big(\frac{1}{2}\int_0^x K(x-y,y)u(x-y,s)(c+d)(y,s)dy - \\ -d(x,t)\int_0^\infty K(x,y)u(y,s)dy + \int_0^\infty F(x,y)u(x+y,s)dy\Big)ds. \end{aligned} \tag{14}$$

By utilizing (5), (6), (8), we consider the second summand in (14) separately. The main idea is to present the "tail" of the infinite integral $\int_X^\infty$ in the form $\int_0^\infty - \int_0^X$ and to use that the first moment of u is equal to zero due to (8):

$$\begin{aligned} \int_0^\infty K(x,y)u(y,t)dy &= \int_0^{X(x)} K(x,y)u(y)dy + \int_{X(x)}^\infty (a(x)y + b(x,y))u(y,t)dy \\ &= \int_0^{X(x)} K(x,y)u(y)dy + \int_{X(x)}^\infty b(x,y)u(y,t)dy - a(x)\int_0^{X(x)} yu(y,t)dy. \end{aligned}$$

Hence,

$$\left|\int_0^\infty K(x,y)u(y,t)dy\right| \leq$$
$$\leq \left[\sup_{0\leq y\leq X(x)} K(x,y) + \sup_{y\geq X(x)} b(x,y) + a(x)X(x)\right]\int_0^\infty |u(y,t)|dy. \quad (15)$$

Using (15) and (6), we obtain from (14):

$$|u(x,t)| \leq \int_0^t \left\{\frac{1}{2}\int_0^x K(x-y,y)|u(x-y,s)|\cdot|c+d|(y,s)dy+\right.$$
$$\left.+G\exp(\lambda x)|d(x,s)|\int_0^\infty |u(y,s)|dy + \int_0^\infty F(x,y)|u(x+y,s)|dy\right\} ds. \quad (16)$$

Let
$$U(q,t) = \int_0^\infty \exp(-qx)|u(x,t)|dx,$$
$$\psi = \max\{|c+d|,|c|,|d|\}, \qquad \Psi(q,t) = \int_0^\infty \exp(-qx)\psi(x,t)dx.$$

Let q be on the real axis. Functions U and Ψ decrease in $q,\ q\geq 0$. Boundness of the values $U(0,t),\Psi(0,t)$ ensures that all partial derivatives in q of U,Ψ are bounded on $q>0$. In addition, the functions U and Ψ are continuous with all their derivatives in q for any fixed $t,\ 0\leq t\leq T$. Since $u(x,t)$ and $\psi(x,t)$ are continuous, then U ans Ψ are continuous together with all their partial derivatives with respect to q for $q>0,\ \ 0\leq t\leq T$.
If we choose $q_0>\max\{\lambda,\mu\}$ and utilize (7), then for $0<q_0\leq q\leq q_1<\infty$ the following inequality takes place

$$\int_0^\infty\int_0^\infty F(x,y)\exp(-qx)|u(x+y,t)|dy =$$
$$= \int_0^\infty |u(x,t)|\int_0^x F(x-y,y)\exp(-qy)dydx \leq AU(0,t). \quad (17)$$

By integrating (16) with weight $\exp(-qx)$ and taking into account (17), (6), we obtain

$$\begin{aligned} U(q,t) \ &\leq\ \int_0^t\{GU(q,s)\Psi(q-\lambda,s) - GU_q(q,s)\Psi(q-\lambda,s)+\\ &+\ G\Psi(q-\lambda,s)U(0,s)+AU(0,s)\}ds. \end{aligned} \quad (18)$$

Our next step is to estimate $U(0,s)$. Let q_2 be the solution to the algebraic equation

$$U(0,t) = U(q_2,t) - q_2 U_q(q_1,t). \tag{19}$$

Due to the decreasing of the function $U(q,t)$ with q, the equation (19) has the only root $q_2 > q_1$. Hence, for $0 \le q \le q_1, \quad 0 \le t \le T$ we obtain

$$U(0,t) \le U(q,t) - QU_q(q,t), \tag{20}$$

where $Q = \sup_{0\le t\le T} q_2(t)$. By substituting (20) into (18) we come to the following inequality

$$U(q,t) \le \int_0^t \{V(q,s)U(q,s) - W(q,s)U_q(q,s)\}\, ds, \quad 0 < q_0 \le q \le q_1 \tag{21}$$

where functions V and W are positive, continuous and have negative first derivative in q. Similarly, by integrating (16) with weight $x\exp(-qx)$, we obtain

$$-U_q(q,t) \le -\int_0^t \frac{\partial}{\partial q} \{V(q,s)U(q,s) - W(q,s)U_q(q,s)\}\, ds, \tag{22}$$

if $0 < q_0 \le q \le q_1$. We choose the value of q_1 sufficiently large to take $\varepsilon \ge Tc_1$ and to obtain $T' = T$. Applying Lemma to (21),(22) we obtain $U(q,t) = 0$ in the region R defined in Lemma. Since $|u(x,t)|$ is continuous, $u(x,t)$=0 for $0 \le t \le T, \quad 0 \le x < \infty$. Consequently, $c = d$. This completes the proof of Theorem.

Remarks

4. The theorem is true for the discrete form of the problem (1),(2) and for the case including source and efflux terms which mathematically means adding $h(x,t) - f(x,t)c(x,t)$ with $f(x,t) \le \text{const} \cdot (1+x)$ to the right hand side of the equation (1).

5. Also, we extend results of Stewart's uniqueness theorem [15] and obtain uniqueness for cases of his existence theorem [14] which are not covered by [15].

6. Our theorem includes as well the important part of the Spouge's conditions ensuring existence [13]. Namely, his conditions on fragmentation, sources and efflux satisfy our uniqueness theorem and the remark 4. In addition, we have the large intersection with Galkin & Dubovski's [8] conditions on the coagulation kernels ensuring existence. These kernels include many unbounded kernels modelling fast interaction of particles with approximately equal masses ($x \approx y$). The following function $K(x,y)$ from (5) can serve as an example:

$$K(x,y) = \alpha(x,y) + \begin{cases} \exp(\nu(2y-x)), & y \le x \\ \exp(\nu(2x-y)), & y \ge x \end{cases}, \quad 0 \le \nu \le \lambda.$$

The function α is bounded as before.

7. For the coagulation kernel $K = xy$ without fragmentation Ernst *et al* [5] and Galkin [7] found exact behaviour of the first moment of solution. Consequently, this case conforms to the condition (8) of the theorem, and we have global uniqueness of solution. Uniqueness theorem for such coagulation model was proved by McLeod [10] for short time interval when the mass conservation law takes place (before gel creating). For the Flory-Stockmayer discrete model of polymerization with $K_{i,j} = (Ai+2)(Aj+2)$ Ziff and Stell [18] found the value of the first moment of solution for all $t \ge 0$. Consequently, in this case we obtain global uniqueness of solution, too.

8. Recently Bruno, Friedman and Reitich [2] considered a special coagulation model. They succeded to prove uniqueness for bounded coagulation kernels only, though their existence theorem allows to concern unbounded ones. Our approach supplements their results and enables to prove uniqueness of solution for kernels from the class (5).

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