Travelling Wave-Like Solutions of the Navier-Stokes and the Related Equations.

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Abstract

We present new family of travelling wave-like solutions to the Navier-Stokes equations of incompressible fluid flows, and other regularized equations of the Euler equations, obtain their trend to the solutions of Euler equations as the viscosity tends to zero and estimate the rate of convergence. We also find a "singularizing effect" of the viscosity term in the Navier-Stokes equations, i.e. we have a local moving blow-up of unbounded solutions with the blow-up's speed depending on viscosity. We demonstrate that if the initial function is the Beltrami flow then the solution of Navier-Stokes equations conserves the Beltrami flow property for all time.

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1 Introduction

We examine the following system of equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu B u \tag{1}$$

$$\text{div } u = 0, \tag{2}$$

$$u(x,0) = u^{(0)}(x), \quad x \in R^n, \ \ n \geq 2, \quad t \in R^1_+ = [0, \infty). \tag{3}$$

If $B = \Delta$ then we obtain Navier-Stokes equations for incompressible fluid flow. In (1) $u = u(x,t) \in R^n$ is an unknown function from $R^n \times R^1_+$ to R^n, usually it conforms to the velocity of fluid at time $t \geq 0$ at a point $x \in R^n, \quad n \geq 2$; p is unknown scalar function of (x,t) which describes pressure. The linear differential operator B is supposed to be expressed by

$$Bu = B_1 u + \lambda(t) u,$$

where $\lambda(t)$ is a scalar time-dependent function and

$$B_1 = \sum_{i_1+i_2+\ \dots\ +i_n \geq 1} a_{i_1 i_2\ \dots\ i_n}(x,t) \left(\frac{\partial}{\partial x_1}\right)^{i_1} \left(\frac{\partial}{\partial x_2}\right)^{i_2} \ \dots\ \left(\frac{\partial}{\partial x_n}\right)^{i_n}.$$

It turns out that the following proposition takes place for the Euler equations for ideal liquid which can be obtained from (1) putting $\nu = 0$ (see [1]).

Theorem 1 *Let $k \in R^n$ be a constant and*

$$u^{(0)}(x) = h(k \cdot x), \quad h : R^1 \mapsto R^n, \quad k \cdot h = c = const. \tag{4}$$

Then the initial value problem for Euler equations has infinitely many travelling wave-like solutions

$$u(x,t) = h\left(k \cdot \left(x + \int_0^t A(s) ds\right) - ct\right) - A(t), \quad p(x,t) = \frac{d}{dt} A(t) \cdot x \tag{5}$$

provided that $A(t) \in R^n$ is a time-dependent function with $A(0) = 0$.

Remark 1 *If we choose $A(t) \equiv 0$ then we obtain the travelling wave*

$$u(x,t) = h(k \cdot x - ct). \tag{6}$$