Convergence to equilibrium for coagulation equation with sources

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In this paper we are concerned with the coagulation equation with sources

$$\frac{\partial c(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy-$$

$$-c(x,t)\int_0^\infty K(x,y)c(y,t)dy + q(x), \quad x \geq 0, t > 0, \tag{1}$$

$$c(x,0) = c_0(x) \geq 0, \quad x \geq 0. \tag{2}$$

Our aim is to reveal some properties of the equilibrium solution and prove convergence of the time-dependent solution to the stationary one. Stationary coagulation equation with sources was studied in [2] where the equation with effluxes was treated. Taking effluxes into consideration essentially helps to construct results. Without efflux term the convergence to equilibrium was not proved before.

1 Properties of the stationary solutions

The stationary form of the equation (1) is

$$\frac{1}{2}\int_0^x K(x-y,y)\bar{c}(x-y)\bar{c}(y)dy - \bar{c}(x)\int_0^\infty K(x,y)\bar{c}(y)dy + q(x) = 0, \quad x \geq 0. \tag{3}$$

Let $\bar{c}(x)$ be its nonnegative measurable solution for which the integrals in (3) are bounded for anu $x \geq 0$. Obviously, for the coagulation kernel $K(x,y)v(x)v(y)$ the solution of (3) is $\bar{c}(x)/v(x)$ for any function $v(x) \geq 0$.

Integrating (3) with the weight x yields

$$\int_0^\infty \int_0^\infty xK(x,y)\bar{c}(x)\bar{c}(y)dxdy = \infty. \tag{4}$$

In fact, otherwise the first and the second summands in (3) which become equal to (4), yield zero, and we come to the contradiction with the positivity of

$$\int_0^\infty xq(x)dx.$$

From (4) we conclude that if $K(x,y) \leq M = \text{const}$ then the first moment of the function $\bar{c}(x)$ is unbounded. From physical point of view this simple result is very natural: a long-time influx of particles in the disperse system brings up the infinite total mass. Nevertheless, the total amount of particles expressed by the zero moment of $\bar{c}(x)$, may be bounded. For the kernels which describe weak coagulation (e.g. $K(x,y) = \exp(-x-y)$), the zero moment can be infinite similarly to the first one. We define the moments of the solution as

$$N_\alpha = \int_0^\infty x^\alpha \bar{c}(x)dx.$$

If we restrict ourselves with solutions $\bar{c}(x)$ with bounded zero and unbounded first moments, then the natural question arises: "When the α-th moment of the equilibrium solution becomes unbounded?" The following theorem gives the answer to this question.

Theorem 1 *Let symmetric nonnegative continuous coagulation kernel be bounded in $L^\infty(R_+^2)$ and nonzero nonnegative function of sources q have bounded first moment. Let there exist at least one nonnegative measurable solution $\bar{c}$ of (3). Then on $\alpha \geq 1/2$ the moments N_α are equal to infinity.*

Remark 1 *The hypothesis of solvability of the equation (3) is essential. Actually, if $K(x,y) = 0$ on $x > 1$ or $y > 1$ and the sources function q is not equal to zero on $x > 2$, then the equation (3) is unsolvable.*

Proof of Theorem 1. Multiplying (3) by x^α and integrating yields

$$\frac{1}{2}\int_0^\infty \int_0^\infty \left[(x+y)^\alpha - x^\alpha - y^\alpha\right] K(x,y)\bar{c}(x)\bar{c}(y)dxdy = -Q_\alpha \tag{5}$$

where

$$Q_\alpha = \int_0^\infty x^\alpha q(x)dx > 0. \tag{6}$$

The following inequality holds for all $x, y \geq 0$:

$$(x+y)^\alpha - x^\alpha - y^\alpha \geq (2^\alpha - 2)x^{\alpha/2}y^{\alpha/2}, \qquad \text{if} \quad 0 \leq \alpha \leq 1, \ \alpha \geq 2. \tag{7}$$

To prove (7) it suffices to note that the minimum of the function

$$\frac{(x+y)^\alpha - x^\alpha - y^\alpha}{x^{\alpha/2}y^{\alpha/2}}$$

is achieved at $x = y$.

We substitute (7) into (5) and obtain

$$2Q_\alpha \leq (2 - 2^\alpha)MN_{\alpha/2}^2, \qquad 0 \leq \alpha \leq 1. \tag{8}$$

Here

$$M = \sup_{0 \leq x,y < \infty} K(x,y),$$

If to assume $N_{1/2} < \infty$ then at $\alpha = 1$ we obtain from (8) the contradiction $Q_1 \leq 0$. This proves the Theorem 1.

Further, we consider the constant case $K(x,y) \equiv 1$. The case $K = \text{const}$ can be transformed onto $K = 1$ by change of variables $\tau = Kt$. We put for convenience $Q = Q_0$ where Q_0 is defined in (6). It is easily to observe that $N_0 = \sqrt{2Q}$. We substitute this correlation into (3):

$$\frac{1}{2}\bar{c} * \bar{c}(x) - \sqrt{2Q}\bar{c}(x) + q(x) = 0. \tag{9}$$

In (9) $\bar{c} * \bar{c}$ means the convolution:

$$\bar{c} * \bar{c}(x) = \int_0^x \bar{c}(x-y)\bar{c}(y)dy.$$

We avail ourselves of the Laplace transform and obtain from (9)

$$\bar{c}(x) = \sqrt{2Q}\sum_{i=1}^\infty \frac{(2i-3)!!q^{[i]}(x)}{(2Q)^i i!}, \tag{10}$$

where

$$(2i-3)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2i-3), \quad (-1)!! = 1,$$

$$q^{[i]} = q * q * \ldots * q \quad \text{(the convolution is used } i-1 \text{ times).}$$

By definition $q^{[0]} = 1$, $q^{[1]} = q$. The expression (10) testifies the nonnegativity and uniqueness of the solution to (9) and allows explicitely find $\bar{c}(x)$ for simple source functions. Let us point out that for $q(x) = \exp(-ax)$ we obtain

$$\bar{c}(x) = \sqrt{2a} \sum_{i=1}^{\infty} \frac{(2^i - 3)!!(ax)^{i-1}}{2^i i!(i-1)!} \mathrm{e}^{-ax} = \sqrt{a/2} \exp\left(-\frac{ax}{2}\right)\left(I_0\left(\frac{ax}{2}\right) - I_1\left(\frac{ax}{2}\right)\right) \tag{11}$$

where I_0, I_1 are modified Bessel functions. Also, the equality (10) allows to conclude that $N_\alpha < \infty$ for all $0 \le \alpha < 1/2$ provided that

$$q(x) \le M_0 \exp(-ax), \qquad M_0 = \text{const.} \tag{12}$$

Really, in this case $Q \le M_0/a$ and from (10) we conclude

$$\bar{c}(x) \le \sqrt{2M_0 a} \sum_{i=1}^{\infty} \frac{(2^i - 3)!!(ax)^{i-1}}{2^i i!(i-1)!} \mathrm{e}^{-ax} \tag{13}$$

By integrating (13) with weight x^α, $0 \le \alpha \le 1$ we obtain

$$N_\alpha = \frac{\sqrt{M_0} a^{-(\alpha+1/2)}}{\sqrt{2\pi}} \cdot \sum_{i=1}^{\infty} \frac{\Gamma(i - 1/2)\Gamma(i+\alpha)}{\Gamma(i+1)\Gamma(i)}$$

where $\Gamma(i)$ is Euler's gamma-function. We have utilized that

$$\Gamma(i + 1/2) = 2^{-i}\sqrt{\pi}(2i-1)!!.$$

Applying the Raabe's test of summation of series ([1], p.273), we find $N_\alpha < \infty$ provided that $\alpha < 1/2$. Consequently, in this case the estimate $\alpha = 1/2$ of Theorem 1 is exact, and we come to the following lemma.

Lemma 1 *Let the conditions of Theorem 1 and (12) hold. Then* $N_\alpha < \infty$ *provided that* $\alpha < 1/2$.

If we consider the discrete stationary coagulation equation

$$\frac{1}{2} \sum_{j=1}^{i-1} \bar{c}_{i-j}\bar{c}_j - \bar{c}_i \sum_{j=1}^{\infty} \bar{c}_j + q_i = 0, \quad i \ge 1$$

with sources $q = (Q, 0, 0, \ldots, 0, \ldots)$, then

$$\bar{c}_i = \sqrt{2Q} \frac{(2i-3)!!}{2^i i!}, \quad i \ge 1.$$

In this case we also have $N_\alpha < \infty$ on $\alpha < 1/2$.

2 Convergence to equilibrium

Theorem 2 *Let conditions of Theorem 1 hold, the coagulation kernel* K *be a constant and the sources function* q *be continuous. Then the solution of the problem (1), (2) converges to equilibrium as* $t \to \infty$ *in* $C[a,b]$ *for all* $0 \le a < b < \infty$. *If, in addition, (12) holds then the convergence takes place in* $L^1[0,\infty)$. *The rate of convergence is proportional to* $\exp\left(-\sqrt{2Q}\ t\right)$.

Proof. As we have already mentioned, we can transform any constant coagulation kernel to the unit one. Hence, put $K = 1$. We denote $f(x,t) = c(x,t) - \bar{c}(x)$. Then for the function f we obtain from (1)–(3):

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2} f * f(x,t) - f(x,t)\int_0^\infty f(y,t)dy + \bar{c} * f(x) -$$

$$-\bar{c}(x)\int_0^\infty f(y,t)dy - f(x,t)\int_0^\infty \bar{c}(y)dy, \tag{14}$$

$$f(x,0) = f_0(x) \overset{\text{def}}{=} c_0(x) - \bar{c}(x). \tag{15}$$

We denote $F(t) = \int_0^\infty f(x,t)dx$. Integrating the equation (1) yields

$$F(t) = \frac{2\sqrt{2Q}}{\left(1 + \frac{2\sqrt{2Q}}{F(0)}\right)\exp(\sqrt{2Q}\ t) - 1}. \tag{16}$$

Hence,

$$F(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{17}$$

Obviously, $F(t) \equiv 0$ provided that $F(0) = 0$. If $F(0) > 0$ then from (16) we obtain

$$0 < F(t) < F(0)\exp(-\sqrt{2q}\ t). \tag{18}$$

For $-\sqrt{2Q} \le F(0) < 0$ we have $-\sqrt{2Q} \le F(t) < 0$ and, in addition,

$$|F(t)| \le 2|F(0)|\exp(-\sqrt{2Q}\ t) \le 2\sqrt{2Q}\ \exp(-\sqrt{2q}\ t). \tag{19}$$

Also, we observe

$$\int_0^t |F(s)|ds \le \frac{2F(0)}{F(0) + 2\sqrt{2Q}} \le 2 \tag{20}$$

provided that $F(0) < 0$.

With the aim to show that $f(x,t) \to 0$ as $t \to \infty$ we consider the linearised equation (14):

$$u_t(x,t) = -F(t)u(x,t) + \bar{c} * u(x,t) - F(t)\bar{c}(x) - \sqrt{2Q}\, u(x,t), \qquad u(x,0) = u_0(x). \tag{21}$$

Employing the Laplace transform gives the solution to (21):

$$u(x,t) = \exp\left(-\sqrt{2Q}\, t - \int_0^t F(s)ds\right)\left\{u_0(x) + u_0(x) * \sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}(x)t^i}{i!} - \right.$$
$$\left. - \int_0^t F(s)\exp\left(\sqrt{2Q}\, s + \int_0^s F(s_1)ds_1\right) \sum_{i=0}^{\infty} \frac{\bar{c}^{[i+1]}(x)(t-s)^i}{i!} ds\right\}. \tag{22}$$

Therefore we use the method of variation of constants to look for solution of (14), (15) in the form

$$f(x,t) = \exp\left(-\sqrt{2Q}\, t - \int_0^t F(s)ds\right)\left\{g(x,t) + g(x,t) * \sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}(x)t^i}{i!} - \right.$$
$$\left. - \int_0^t b(s) \sum_{i=0}^{\infty} \frac{\bar{c}^{[i+1]}(x)(t-s)^i}{i!} ds\right\}, \tag{23}$$

where

$$b(t) = F(t)\exp\left(\sqrt{2Q}\, t + \int_0^t F(s)ds\right).$$

From (18), (19) we conclude

$$b(t) \le F(0)\exp(F(0)/\sqrt{2Q}) \quad \text{if} \quad F(0) > 0; \tag{24}$$

$$|b(t)| \le 2|F(0)| \quad \text{if} \quad F(0) \le 0. \tag{25}$$

Substituting (23) into (14) yields

$$g_t + g_t * \sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}t^i}{i!} = \frac{1}{2}\exp\left(-\sqrt{2Q}\, t - \int_0^t F(s)ds\right) \cdot$$
$$\cdot \left\{g^{[2]} + g^{[2]} * \left(\sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}t^i}{i!}\right)^{[2]} + 2g^{[2]} * \sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}t^i}{i!} - \right.$$

$$-2g*\left(\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)*\int_0^t b(s)\sum_{i=0}^{\infty}\frac{\bar{c}^{[i+1]}(t-s)^i}{i!}ds-$$

$$-2g*\int_0^t b(s)\sum_{i=0}^{\infty}\frac{\bar{c}^{[i+1]}(t-s)^i}{i!}ds+$$

$$+\left(\int_0^t b(s)\sum_{i=0}^{\infty}\frac{\bar{c}^{[i+1]}(t-s)^i}{i!}ds\right)^{[2]}\Bigg\},\qquad g(x,0)=f_0(x). \tag{26}$$

With the aim of the Laplace transform we conclude from (26) that the function g satisfies the equation

$$g_t=\frac{1}{2}\exp\left(-\sqrt{2Q}\;t-\int_0^t F(s)ds\right)\cdot\Bigg\{g^{[2]}+g^{[2]}*\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}+$$

$$+\left(\bar{c}^{[2]}+\bar{c}^{[2]}*\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)*\left(\int_0^t b(s)ds+\sum_{i=1}^{\infty}\frac{(-\bar{c})^{[i]}}{i!}\int_0^t b(s)s^i ds\right)^{[2]}-$$

$$-2g*\int_0^t b(s)\sum_{i=0}^{\infty}\frac{\bar{c}^{[i+1]}(t-s)^i}{i!}ds\Bigg\},\qquad g(x,0)=f_0(x). \tag{27}$$

We write (27) in the integral form, then estimate g and $-g$ with (20), (24) and (25) taken into account, and finally establish the inequality

$$|g|_t\le\frac{1}{2}\exp\left(2-\sqrt{2Q}\right)\cdot\Bigg\{|g|^{[2]}+|g|^{[2]}*\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}+$$

$$+A^2*\left(\left(\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)^{[3]}+\left(\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)^{[2]}\right)+2A|g|*\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\Bigg\}. \tag{28}$$

In (28) the constant A is equal to one of the upper estimate of $|b(t)|$ in dependence on the sign of $F(0)$.

Let us fix $m>0$. For any $\varepsilon>0$ we can find constants M and M_1 such that

$$\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}(x)t^i}{i!}\le M\mathrm{e}^{\varepsilon t},\quad 0\le x\le m,\quad t\ge 0; \tag{29}$$

$$1*\left(\left(\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)^{[3]}+\left(\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)^{[2]}\right)\le M_1\mathrm{e}^{\varepsilon t},\quad 0\le x\le m,\quad t\ge 0. \tag{30}$$

Hence, $|g| < h$ on $0 \le x \le m,\ t > 0,$ where the function h satisfies the equation

$$h_t(x,t) = \frac{1}{2}\exp(2-(\sqrt{2Q}-\varepsilon)t)\cdot\left(h^{[2]} + h^{[2]} * M + A^2M_1 + 2AM * h\right), \quad (31)$$

$$h(x,0) = h_0 = \text{const} > \sup_{0\le x\le m}|f_0(x)|. \quad (32)$$

Let us note that $h(x,t)$ increases in x for all $t > 0$. Actually, since $h_0 = \text{const}$ then from (31) $h_t(x,t) > h_t(x_1,t)$ for $x > x_1,\quad t \ge 0$. Hence, $h^{[2]}$ increases in x, too, and, consequently,

$$M * h^{[2]}(x,t) \le Mmh^{[2]}(x,t), \quad M * h(x,t) \le Mmh(x,t)$$

for $0 \le x \le m,\quad t \ge 0$. We substitute these expressions into (31) and establish that $h(x,t) < H(x,t)$ for $0 \le x \le m,\ t \ge 0,$ if

$$H_t(x,t) = \frac{1}{2}\exp(2-(\sqrt{2Q}-\varepsilon)t)\cdot\left(H^{[2]}(x,t)(1+Mm) + H(x,t)(1+2AMm)\right),$$

$$H(x,0) = H_0 = \text{const} > \max\{h_0, A^2M\}.$$

We solve this equation and obtain

$$H(x,t) = H_0E(t)\exp\left(H_0x(E(t)-1)\frac{1+Mm}{1+2AMm}\right) \quad (33)$$

where

$$E(t) = \exp\left((1-\exp(-(\sqrt{2Q}-\varepsilon)t))\frac{\mathrm{e}^2(1+2AMm)}{2(\sqrt{2Q}-\varepsilon)}\right) \le$$

$$\le \exp\left(\frac{\mathrm{e}^2(1+2AMm)}{2(\sqrt{2Q}-\varepsilon)}\right) = E_0.$$

Finally, from (33) we obtain boundedness of $g(x,t)$:

$$|g(x,t)| \le H_0E_0\exp\left(H_0m(E_0-1)\frac{1+Mx_0}{1+2AMm}\right) = G,\ 0 \le x \le m,\ t \ge 0. \quad (34)$$

Now we substitute (20), (29), (30) and (34) into (23) and conclude that $c(x,t)$ tends to $\bar{c}(x)$ as $t \to \infty$ uniformly with respect to $x \in [0,m]$:

$$|c(x,t) - \bar{c}(x)| \le \exp(2-\sqrt{2Q}\ t)\left(G + GMm\mathrm{e}^{\varepsilon t} + AM\mathrm{e}^{\varepsilon t}\right) \le M_2\mathrm{e}^{-(\sqrt{2Q}-\varepsilon)t}, \quad (35)$$

$$0 \le x \le m, \; t \ge 0.$$

We should emphasize that the constants G and M depend on m and ε. This proves convergence in the space $C[a,b]$ for any $0 \le a < b < \infty$.

To prove convergence in the space $L^1[0,\infty)$ we note that

$$\int_0^\infty |c(x,t)-\bar{c}(x)|dx = \int_0^m |c(x,t)-\bar{c}(x)|dx + \int_m^\infty |c(x,t)-\bar{c}(x)|dx \le$$

$$\le M_2 m \mathrm{e}^{-(\sqrt{2Q}-\varepsilon)t} + \int_m^\infty c(x,t)dx + \int_m^\infty \bar{c}(x)dx. \tag{36}$$

Let us fix $\epsilon > 0$ and pick up $m \ge (N_\alpha/\epsilon)^{1/\alpha}$, $0 < \alpha < 1/2$. Then

$$\int_m^\infty \bar{c}(x)dx \le \epsilon. \tag{37}$$

Really, to obtain (37) we employ Lemma 1 and the inequality for "tails" of integrals (cf.[3])

$$\int_m^\infty \phi(x)dx \le \frac{1}{\psi(m)} \int_0^\infty \phi(x)\psi(x)dx, \quad m > 0,$$

which is true for $\phi(x) \ge 0$ and nondecreasing $\psi(x) > 0$. Since $F(t) \to 0 \quad t \to \infty$ (see (17)) and (35) is valid, then there exists t_0 such that for all $t > t_0$

$$\int_m^\infty c(x,t)dx = \int_0^\infty c(x,t)dx - \int_0^m c(x,t)dx = \int_m^\infty \bar{c}(x)dx + \delta(t) \tag{38}$$

where $\delta(t) \le \epsilon$ for all $t > t_0$. Inserting (37) and (38) into (36) yields

$$\int_0^\infty |c(x,t)-\bar{c}(x)|dx \le M_2 m \mathrm{e}^{-(\sqrt{2Q}-\varepsilon)t} + 3\epsilon. \tag{39}$$

(35) and (39) prove Theorem 2.

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