Reconstruction of source function for inverse transport problem in a slab

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Abstract

The paper deals with a numerical solution of the inverse stationary transport problem in a slab. More precisely, our aim is to reveal a source function on the basis of boundary observation (Problem 1) or on the basis of internal observation (Problem 2). The operator for Problem 1 has, generally, nonzero kernel, and to garantee unique solvability of inverse problem, we select some special classes of functions with zero kernel. We derive iterative algorithms depending on these special classes. Then we present the results on numerical solution at different conditions and some specific examples.

1 Main notions and statements

We are concerned with the following stationary one-velocity transport problem in a slab $0 \le z \le H$ [1, 2]:

$$A\phi \overset{def}{=} \mu \frac{\partial \phi(\mu, z)}{\partial z} + \phi(\mu, z) - \frac{b(z)}{2} \int_{-1}^{1} p(\mu, \mu')\phi(\mu', z)d\mu' = f(\mu, z), \qquad (1)$$

$$0 < z < H,\ -1 \le \mu \le 1, \quad \phi(\mu, z)|_{\Gamma_-} = \phi_{(\Gamma_-)} \overset{\mathrm{def}}{=} (\phi^{(1)}_{(\Gamma_-)}(\mu), \phi^{(2)}_{(\Gamma_-)}(\mu)),$$

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where $0 \le b(z) \le b_1 = \text{const} < 1,\ H < \infty$, and the set Γ_- corresponds to the incoming flow in the slab $0 \le z \le H$:

$$\Gamma_- = \Big\{(\mu, z):\ (\mu \in [0,1],\ z = 0) \cup (\mu \in [-1,0],\ z = H)\Big\}.$$

The phase function $p(\mu, \mu') \ge 0$ satisfies the condition

$$\frac{1}{2}\int_{-1}^{1} p(\mu, \mu')d\mu' = 1 \quad \forall \mu \in [-1, 1]. \tag{2}$$

If we deal with isotropic scattering, then $p(\mu, \mu') \equiv 1$.

Let us define the following sets:

$$X = \Big\{(\mu, z):\ \mu \in [-1,1],\ z \in [0, H]\Big\},$$
$$\Gamma_+ = \Big\{(\mu, z):\ (\mu \in [-1,0],\ z = 0) \cup (\mu \in [0,1],\ z = H)\Big\}$$

and introduce Hilbert spaces $H_2^1,\ L_{2,-},\ L_{2,+}$,

$$H_2^1 = \Big\{\phi:\ \phi \in L_2,\ \|\phi\|_{H_2^1} = \Big(\|\phi\|_{L_2}^2 + \Big\|\mu\frac{\partial \phi}{\partial z}\Big\|_{L_2}^2\Big)^{1/2}\Big\},$$
$$L_{2,-} = \Big\{\gamma_{(-)}(\mu) \stackrel{\text{def}}{=} (\gamma_{(-)}^{(1)}(\mu),\ \gamma_{(-)}^{(2)}(\mu))\Big\} \text{ with}$$
$$\|\gamma_{(-)}\|_{L_{2,-}} = \Big[\int_0^1 \mu|\gamma_{(-)}^{(1)}(\mu)|^2 d\mu + \int_{-1}^{0} \mu|\gamma_{(-)}^{(2)}(\mu)|^2 d\mu\Big]^{1/2},$$
$$L_{2,+} = \Big\{\gamma_{(+)}(\mu) = (\gamma_{(+)}^{(1)}(\mu), \gamma_{(+)}^{(2)}(\mu)) \text{ with}$$
$$\|\gamma_{(+)}\|_{L_{2,+}} = \Big[\int_{-1}^{0} \mu|\gamma_{(+)}^{(1)}(\mu)|^2 d\mu + \int_{1}^{0} \mu|\gamma_{(+)}^{(2)}(\mu)|^2 d\mu\Big]^{1/2}.$$

The components $\gamma_{(-)}^{(1)}(\mu)$ and $\gamma_{(+)}^{(2)}(\mu)$ are defined for $\mu > 0$, and the components $\gamma_{(-)}^{(2)}(\mu)$ and $\gamma_{(+)}^{(1)}(\mu)$ – for $\mu<0$.

We consider some subsets $X_c \subset X$ and $X_{obs} \subset X$ to specify the areas of reconstructing of source functions and the areas of observations, respectively. The following subspaces are also used below:

$$L_2^{(obs)} = \{f : f \in L_2(X),\ f \equiv 0 \text{ in } X \backslash X_{obs}\},$$
$$L_2^{(c)} = \{f : f \in L_2(X),\ f \equiv 0 \text{ in } X \backslash X_c\}.$$

If ϕ is a solution of (1), then

$$\phi(\mu,z)=\begin{cases}\phi^{(1)}_{(\Gamma_-)}(\mu)e^{-\frac{z}{\mu}}+\displaystyle\int_0^z e^{-\frac{z-z'}{\mu}}F(\mu,z')\frac{dz'}{\mu},\ \mu>0,\\ \phi^{(2)}_{(\Gamma_-)}(\mu)e^{\frac{(H-z)}{\mu}}-\displaystyle\int_z^H e^{-\frac{z'-z}{|\mu|}}F(\mu,z')\frac{dz'}{\mu},\ \mu<0,\end{cases} \tag{3}$$

where

$$F(\mu,z)\stackrel{\mathrm{def}}{=}\frac{1}{2}b(z)\int_{-1}^{1}p(\mu,\mu')\phi(\mu,z)d\mu+f(\mu,z).$$

The following assertion is true [3]:

Theorem 1 *If $f\in L_2$ and $\phi_{(\Gamma)}\in L_{2,-}$, then:*

(1) there exists a unique function $\phi\in H^1_2$ which is the solution to problem 1;

(2) the function ϕ satisfies equation (1) almost everywhere in X, and boundary condition for almost all μ;

(3) the following estimates hold:

$$C[\|f\|_{L_2}+\|\phi_{(\Gamma_-)}\|_{L_{2,-}}]\le\|\phi\|_{H^1_2}\le\tilde C[\|f\|_{L_2}+\|\phi_{(\Gamma_-)}\|_{L_{2,-}}],\quad C,\tilde C>0,$$

where the constants C and $\tilde C$ are independent of ϕ, f, and $\phi_{(\Gamma_-)}$.

Below we consider two inverse boundary value problems: the problem with "surface observation" and the problem with an "internal observation". Let us formulate them.

We consider the following boundary value problem

$$A\phi=f+m_c v \text{ in } X,\ \phi=\phi_{(\Gamma_-)} \text{ on } \Gamma_{(-)},$$

where m_c is the characteristic function of subset $X_c\subset X$, $\mathrm{mes}(X_c)\neq 0$: $m_c(\mu,z)=1$ on X_c and $m_c(\mu,z)=0$ on $X\setminus X_c$. Also, we assume $f\in L_2(X)$, $v\in L_2(X_c)$.

We point out that $v\equiv 0$ on $X\backslash X_c$.

Let us assume that the function v ("a control function") is unknown while we know an observation function $\phi_{obs}\in L_{2,+}$ given on an observation subset ${\Gamma_+}^{(obs)}\subset\Gamma_+$, $\mathrm{mes}({\Gamma_+}^{(obs)})\neq 0$. We set

$$m_{obs}\stackrel{\mathrm{def}}{=}\begin{cases}1 \text{ on } {\Gamma_+}^{(obs)};\\ 0 \text{ on } \Gamma_+\backslash{\Gamma_+}^{(obs)}.\end{cases}$$

Let $\phi^{(0)}$ be a solution of the problem

$$A\phi^{(0)} = f \text{ in } X, \quad \phi^{(0)} = \phi_{(\Gamma_-)} \text{ on } \Gamma_-.$$

Then we can formulate the following inverse problem for recovering the source function v based on the boundary observations: *for given $f \in L_2(X), \phi_{obs} \in L_{2,+}$ find $\phi_1 \in H_2^1,\ v \in L_2^{(c)}$ such that*

$$A\,\phi_1 = f + m_c\,v \text{ in } X, \quad \phi_1 = \phi^{(0)} \text{ on } \Gamma_-, \tag{4}$$

and v yields minimum of the functional

$$\alpha||m_c\,v||^2_{L_2} + ||m_{obs}(\phi_1 - \phi_{obs})||^2_{L_{2,+}},\ \alpha = \text{const} \geq 0.$$

Now we can reformulate (4) as the following inverse problem for the function $\phi = \phi_1 - \phi^{(0)}$ (Problem 1):
for given $\phi^{(0)} \in H_2^1,\ \phi_{obs} \in L_{2,+}$ find $\phi \in H_2^1,\ v \in L_2(X)$ such that

$$A\phi = m_c\,v \text{ in } X,\ \phi = 0 \text{ on } \Gamma_-,\ \inf_v J_1(\phi, v), \tag{5}$$

where

$$J_1(\phi, v) = \alpha||m_c\,v||^2_{L_2} + ||m_{obs}(\phi - (\phi_{obs} - \phi^{(0)}))||_{L_{2,+}}, \tag{6}$$

$$\alpha = \text{const} \geq 0, \phi_{obs} \in L_{2,+}.$$

This replacement is performed to zero values of ϕ on Γ_- and to remove the term f in the right-hand side of the equation (4). Also, from this point we can treat the functions ϕ and v as functions that can change their sign, and we do not need to consider the transport problem in the nonnegative cone. It is worth pointing out that in the last case some of the assertions below can be proved easier.

Hereafter we set $\phi_{obs} \equiv 0$ on $\Gamma_+ \backslash {\Gamma_+}^{(obs)}$ in the inverse problem (5).

Let ϕ_{obs} is given on some subset $X_{obs} \subset X,\ \text{mes}(X_{obs}) \neq 0$. Let, as before, m_{obs} be the characteristic function of X_{obs}. If we have internal observation of function ϕ, then the inverse problem (Problem 2) can be formulated as follows:
for given $\phi^{(0)} \in H_2^1,\ \phi_{obs} \in L_2(X)$ find $\phi \in H_2^1,\ v \in L_2(X_c)$ such that

$$A\phi = m_c\,v \text{ in } X,\ \phi = 0 \text{ on } \Gamma_-, \quad \inf_v J_2(\phi, v), \tag{7}$$

where

$$J_2(\phi, v) = \alpha||m_c\, v||^2_{L_2} + ||m_{obs}(\phi - (\phi_{obs} - \phi^{(0)}))||^2_{L_2},$$
$$\alpha = \text{const} \geq 0,\ \ \phi_{obs} \in L_2(X_{obs}),$$

and we assume hereafter that $\phi_{obs} \equiv 0$ on $X \backslash X_{obs},\ v \equiv 0$ on $X \backslash X_c$. Our aim below is investigating inverse problems (5), (7).

If we consider the problem

$$A\phi = v \text{ in } X, \quad \phi = 0 \text{ on } \Gamma_-, \tag{8}$$

then we can represent its solution in the following form:

$$\phi = A^{-1}\, v, \tag{9}$$

where $A^{-1} :\ L_2(X) \mapsto H_2^1$.

We need also to introduce in H_2^1 the trace operator

$$P_{(+)}\phi \stackrel{\text{def}}{=} \phi|_{\Gamma_+}, \quad P_{(+)} : H_2^1 \mapsto L_{2,+}$$

and resolution operators $B,\ B_{oc}$, which map the source function v to the trace of the solution ϕ of problem (8):

$$B \stackrel{\text{def}}{=} P_{(+)}A^{-1} :\ L_2(X) \mapsto L_{2,+},$$
$$B_{oc} \stackrel{\text{def}}{=} m_{obs}Bm_c : L_2(X) \mapsto L_{2,+}.$$

2 Sufficient conditions for reconstructing source functions from boundary observations

An important question is if it is possible to reconstruct uniquely a source function $v(\mu, z)$ on the basis of observation data $\phi|_{\Gamma_+}$. A closely related problem is the problem if the kernel of operator B is trivial. In general, $\text{Ker}B \neq \{0\}$. Indeed, if we consider a smooth function ϕ with a compact support over z, then $Bv = 0$ with $v = A\phi \neq 0$. Hence, in this case $v \in \text{Ker}B$.

Our nearest aim is to select some classes of functions such that they have no intersection with $\text{Ker}\, B$. First, we introduce the set U_1 containing all functions of the form

$$v(\mu, z) = \begin{cases} e^{-(H-z)/\mu}\, v_1(\mu),\ \ \mu > 0, \\ e^{z/\mu}\, v_2(\mu),\ \ \mu < 0, \end{cases}$$

where $(v_1, v_2) \in L_{2,+}$.

Lemma 1 *[4]* $U_1 \cap \operatorname{Ker} B = \{0\}$.

The lemma below claims that isotropic source function ($v = v(z)$) can, theoretically, be reconstructed exactly (i.e., it is reconstructable). So, it is a sufficient condition for unique solvability of equation $Bv = \phi|_{\Gamma_+}$ in the class of isotropic functions.

Lemma 2 *[4] Let $U_2 = \{v : \ v = v(z)\}$ and $p(\mu, \mu') = 1$. Let $\phi = Bv$. Then there is no $v_1(z)$ such that the corresponding solution $\phi_1 = Bv_1$ has the same trace on Γ_+, i.e. for all $v_1(z) \neq v(z)$ we have $Bv_1|_{\Gamma_+} \neq \phi|_{\Gamma_+}$. By other words,*

$$\operatorname{Ker} B \cap U_2 = \{0\}.$$

Remark. Since, in general, $\operatorname{Ker} B \neq \{0\}$ then there exists another non-isotropic source function $v_1(\mu, z)$ dependent of μ with $Bv_1 = \phi|_{\Gamma_+}$.

Let us look for other classes of reconstructable source functions having no intersection with $\operatorname{Ker} B$ at $p(\mu, \mu') \equiv 1$.

Let for some integer $n \geq 0$ and real numbers $\{a_i\}_{i=0}^n$

$$v(\mu, z) = \sum_{i=0}^{n} a_i \mu^i v_i^{(i)}(z), \tag{10}$$

where $f^{(i)}$ means the i-th derivative of f, and for all $i,\ 0 \leq i \leq n$,

$$v_i \in C^{(i)}[0, H],\ v_i^{(k)}(0) = v_i^{(k)}(H) = 0,\ 0 \leq k \leq i.$$

If there is a nonzero solution of (1) $\phi(\mu, z)$ such that $\phi|_{\Gamma_+} = \phi|_{\Gamma_-} \equiv 0$, then, similarly (3), we obtain

$$\phi(\mu, z) = \frac{e^{-z/\mu}}{\mu} \int_0^z e^{y/\mu} \Big[\sum_{i=0}^{n} a_i \mu^i v_i^{(i)}(y) + \frac{1}{2} b(y) \int_{-1}^{1} \phi(\mu', y) d\mu' \Big] dy. \tag{11}$$

Integration by parts gives us the equality

$$\int_0^z e^{y/\mu} \mu^i v_i^{(i)}(y) dy = \sum_{k=1}^{i} e^{z/\mu} \mu^{i-k+1} (-1)^{k-1} v_i^{(i-k)}(z) + (-1)^i \int_0^z e^{y/\mu} v_i(y) dy. \tag{12}$$

At $z = 0$ and $z = H$ the first summand, expressed by the sum, is equal to zero because of the above assumptions on v_i. So, substituting (12) in (11) at $z = H$ yields for all $-1 \le \mu \le 1$:

$$0 = \phi(\mu, H) = \frac{e^{-H/\mu}}{\mu} \int_0^H e^{y/\mu} \Big[\sum_{i=0}^{n} a_i (-1)^i v_i(y) + \frac{1}{2} b(y) \int_{-1}^{1} \phi(\mu', y) d\mu' \Big] dy.$$

The case $\mu = 0$ is also included (if we treat $\mu = 0$ as the limit $\mu \to 0$) due to the following correlation:

$$\lim_{\mu \to 0} \int_0^z \frac{e^{-(z-y)/\mu}}{\mu} f(y) dy = \lim_{\mu \to 0} \lim_{\Delta z \to 0} \int_{z - \Delta z}^{z} \frac{e^{-(z-y)/\mu}}{\mu} f(y) dy = f(z).$$

Since the contents of square brackets depends only on y and the integral is equal to zero for all μ, then also [5]

$$\sum_{i=0}^{n} (-1)^i a_i v_i(y) + \frac{1}{2} b(y) \int_{-1}^{1} \phi(\mu', y) d\mu' = 0, \ \ 0 \le y \le H. \tag{13}$$

We substitute (12) and (13) in (11) and obtain

$$\phi(\mu, z) = \sum_{i=0}^{n} a_i \sum_{k=1}^{i} (-1)^{k-1} \mu^{i-k} v_i^{(i-k)}(z). \tag{14}$$

Now we substitute this expression for ϕ in (13), and after integration over μ we finally obtain a correlation for v_i:

$$\sum_{i=0}^{n} a_i \Big[(-1)^i v_i(z) + \frac{1}{2} b(z) \sum_{k=1}^{i} \frac{(-1)^{k-1} + (-1)^{i-1}}{i - k + 1} v_i^{(i-k)}(z) \Big] = 0, \ \ 0 \le z \le H. \tag{15}$$

We are now in a position to prove the following lemma.

Lemma 3 *Let there is a function $f(z) \in C^{(n)}[0, H]$ such that in (10) $v_i(z) = f^{(\alpha_i)}(z)$, $0 \le i \le n$, where integer numbers $\alpha_i \ge 0$ and*

$$f^{(k)}(0) = f^{(k)}(H) \ \textit{for} \ 0 \le \min_i \alpha_i \le k \le \max_i \alpha_i < \infty. \tag{16}$$

Let us denote the corresponding class of v as U_3. Then

$$\operatorname{Ker} B \cap U_3 = \{0\}.$$

Proof. Substituting these function $v_i = f^{(\alpha_i)}$ into (15) yiels us a linear ordinary differential equation with respect to $f(z)$. Recalling (16) we obtain the desired equality $f(z) \equiv 0$. Consequently, $v \equiv 0$. This proves the lemma.
Corollary 1. If $\alpha_i = n - i$ then the source function is multiplicative, i.e.,

$$v(\mu, z) = h(\mu) f^{(n)}, \quad h(\mu) = \sum_{i=0}^{n} a_i \mu^i. \tag{17}$$

This class of functions v we denote by $U_4,\ U_4 \subset U_3$. Consequently,

$$\operatorname{Ker} B \cap U_4 = \{0\}. \ \Box$$

Corollary 2. If $\alpha_i = i$ then the source function takes the form

$$v(\mu, z) = \sum_{i=0}^{n} a_i \mu^i f^{(i)}. \tag{18}$$

This class of functions v we denote by $U_5,\ U_5 \subset U_3$. Consequently,

$$\operatorname{Ker} B \cap U_5 = \{0\}. \ \Box$$

Observing the results of this section, we see that there are source functions (from classes $U_1,\ U_2,\ U_3,\ U_4,\ U_5$) such that all other source functions from the class under consideration yield other observation data from $L_{2,+}$. Consequently, on the basis of such observation data we can uniquely (in the corresponding class) reconstruct source functions.

3 Solvability of inverse problem

To minimize functional J_1 defined in (6) we consider its variation δJ_1 and equal it to zero. So, we obtain

$$\delta J_1 = J_1(2\alpha(m_c v, \delta v) + 2(m_{obs}(\phi - \phi_{obs}), \delta\phi)_{L_{2,+}},$$

where $(.,.)$ means the scalar product in $L_2(X)$. Hence,

$$\alpha(m_c v, \delta v) + (m_{obs}(\phi - \phi_{obs}), \delta\phi)_{L_{2,+}} = 0. \tag{19}$$

Since

$$A^*q(\mu,z) = -\mu\frac{\partial q(\mu,z)}{\partial z} + q(\mu,z) - \frac{1}{2}b(z)\int_{-1}^{1} p(\mu,\mu')q(\mu',z)d\mu',$$

then integration by parts yields

$$(A^*q,\delta\phi) = -(q,\delta\phi)_{L_{2,+}} + (q,\delta v).$$

We set $A^*q = 0$ and obtain

$$(q,\delta\phi)_{L_{2,+}} = (q,\delta v).$$

If, in addition, we impose on q the boundary condition $q|_{\Gamma_+} = m_{obs}(\phi - \phi_{obs})$, then we finally obtain from (19) the basic correlation

$$(\alpha\, m_c v + m_c q, \delta v) = 0. \tag{20}$$

In general, δv can be an arbitrary function from $L_2(X)$. Hence, we arrive at the control equation $\alpha\, m_c v + m_c q = 0$.

So, Problem 1 (equation (5)) can be reformulated as follows:
Problem 1. For given $\phi^{(0)} \in H_2^1$, $m_{obs}\phi_{obs} \in L_{2,+}$ find $\phi \in H_2^1$, $q \in H_2^1$, $v \in L_2^{(c)}$ such that

$$A\phi = m_c v \text{ in } X,\ \phi{=}\ 0 \text{ on } \Gamma_-, \tag{21}$$

$$A^*q = 0 \text{ in } X,\ q = m_{obs}P_{(+)}(\phi - (\phi_{obs} - \phi^{(0)})) \text{ on } \Gamma_+, \tag{22}$$

$$\alpha v + q = 0 \text{ in } X_c,\ v \equiv 0 \text{ in } X\backslash X_c.\square \tag{23}$$

Applying the results of the previous section we can observe that the solution of (21)–(23) cannot, in general, yield a true source function $v(\mu,z)$ because of nonzero kernel $\operatorname{Ker} B$. So, let us restrict our consideration to the classes U_1, U_2, and U_4.

First, we consider class U_2 of isotropic functions. In this case deviation δv is independent of μ, and, hence, control equation (23) does not follow from correlation (20) because μ-independent functions δv are not dense in $L_2(X)$. Writing the scalar product in $L_2(X)$ as integrals, we obtain from (20) the following identity:

$$\int_0^H \Big(2\alpha m_c v(z) + m_c\int_{-1}^{1} q(\mu,z)d\mu\Big)\delta v(z)dz = 0.$$

Since $\{\delta v(z)\}$ is dense in $L_2[0,H]$, then control equation (23) can be replaced by

$$\alpha m_c v(z) + \frac{1}{2} m_c \int_{-1}^{1} q(\mu,z) d\mu = 0. \tag{24}$$

Similarly, for class U_1 we obtain from (20)

$$\int_0^1 \Big[\frac{1}{2}\alpha\mu m_c v_1(\mu)\Big(1-e^{-2H/\mu}\Big) + m_c \int_0^H e^{-(H-z)/\mu} q(\mu,z)dz\Big]\delta v_1(\mu)d\mu$$
$$+\int_{-1}^0 \Big[\frac{1}{2}\alpha\mu m_c v_2(\mu)\Big(e^{2H/\mu}-1\Big) + m_c \int_0^H e^{z/\mu} q(\mu,z)dz\Big]\delta v_2(\mu)d\mu = 0.$$

Consequently, for reconstruction source functions from class U_1, control equation (23) should be replaced by

$$\frac{1}{2}\alpha\mu m_c v_1(\mu)\Big(1-e^{-2H/\mu}\Big) + \int_0^H e^{-(H-z)/\mu} m_c q(\mu,z)dz = 0,\ \mu > 0,$$
$$\frac{1}{2}\alpha\mu m_c v_2(\mu)\Big(e^{2H/\mu}-1\Big) + \int_0^H e^{z/\mu} m_c q(\mu,z)dz,\ \mu < 0. \tag{25}$$

These reasonings can be applied to class U_4, too. If we replace $v(\mu,z)$ by $h(\mu)v(z)$ and assume that μ-dependent multiplier $h(\mu)$ is prescribed, then control equation (23) takes the form

$$\alpha v(z)\int_{-1}^{1} m_c h^2(\mu)d\mu + \int_{-1}^{1} m_c h(\mu) q(\mu,z)d\mu = 0,\ 0<z<H,\ v(0)=v(H)=0. \tag{26}$$

Othewise, if the function $f^{(n)}(z)$ is known, then we arrive at the following control equation for computing $h(\mu)$:

$$\alpha h(\mu)\int_0^H m_c\Big(f^{(n)}(z)\Big)^2 dz + \int_0^H m_c q(\mu,z) f^{(n)}(z)dz, \quad h(\mu) = \sum_{i=0}^{n} a_i \mu^i. \tag{27}$$

At this point we finish the theoretical analysis of Problem 1.

Analoguously to deriving equations (21)–(23), the second inverse problem can be reformulated as follows:

Problem 2. For given $m_{obs}(\phi_{obs} - \phi^{(0)}) \in L_2(X)$, find $v \in L_2^{(c)}$ such that

$$A\phi = m_c v \text{ in } X,\ \phi = 0 \text{ on } \Gamma_-, \tag{28}$$

$$A^*q = m_{obs}(\phi - (\phi_{obs} - \phi^{(0)})) \text{ in } X,\ q = 0 \text{ on } \Gamma_+, \tag{29}$$

$$\alpha m_c v + m_c q = 0 \text{ in } X.\Box \tag{30}$$

For Problems 1, 2 the following statements hold [4]:

Lemma 4 *If $\alpha>0$, then inverse problem (21)–(23) has a unique solution for any $m_{obs}P_+(\phi_{obs} - \phi^{(0)}) \in L_{2,+}$.*

If $\alpha = 0$, $m_{obs}P_{(+)}(\phi_{obs} - \phi^{(0)})$ is in the range $R(B_{oc})$ of $B_{oc} = m_{obs}Bm_c$ and $v \in U_1$ or U_2, then this problem also has a unique solution.

Lemma 5 *If $\alpha > 0$, then problem (28)–(30) has a unique solution for any $m_{obs}(\phi_{obs} - \phi^{(0)}) \in L_2$. If $\alpha = 0$, $m_{obs}(\phi_{obs} - \phi^{(0)}) \in R(m_{obs}A^{-1}m_c)$ and $X_{obs} \supseteq X_c$, then this problem also has a unique solution.*

Similar lemmas for the case $\alpha = 0$ can be also formulated for classes U_3, U_4, U_5.

To construct an approximate solution of (21)–(23), the following algorithm can be applied:

$$A\phi_n = m_c v_n \text{ in } X,\ \phi_n = 0 \text{ on } \Gamma_-, \tag{31}$$

$$A^*q_n = 0 \text{ in } X,\ \ q_n = m_{obs}P_+(\phi_n - (\phi_{obs} - \phi^{(0)})) \text{ on } \Gamma_+, \tag{32}$$

$$v_{n+1} = v_n - \tau_1(\alpha v_n + q_n) \text{ in } X_c,\ v_{n+1} \equiv 0 \text{ in } X\backslash X_c,\ \ n = 0, 1, ..., \tag{33}$$

where $v_0 \in L_2^{(c)}$ and $\tau_1 = 2/(2\alpha + \gamma_1)$ with

$$\gamma_1 = \left[(1 - b_1)(1 + \sqrt{1 - b_1}\coth(H\sqrt{1 - b_1})\right]^{-1}.$$

Taking into account the properties of our problem and the results of iterative processes theory, the following result can be obtained:

Lemma 6 *[4] If $\alpha > 0$ and $\tau_1 = 2/(2\alpha + \gamma_1)$, then algorithm (31)–(33) converges, and the following estimate holds:*

$$\|\phi - \phi_n\|_{H_2^1} + \|q - q_n\|_{H_2^1} + \|m_c(v - v_n)\|_{L_2} \le C\left(\frac{\gamma_1}{2\alpha + \gamma_1}\right)^n \to 0,\ \ n \to \infty,$$

where $C = C(v_0, \phi_{obs}, \phi^{(0)}) = \text{const} > 0$.

The algorithm for Problem 2 can be written as follows:

$$
\begin{aligned}
&A\phi_n = m_c v_n \text{ in } X,\ \phi_n = 0 \text{ on } \Gamma_-,\\
&A^* q_n = m_{obs}(\phi_n - (\phi_{obs} - \phi^{(0)})) \text{ in } X,\ q = 0 \text{ on } \Gamma_+,\\
&v_{n+1} = v_n - \tau_2(\alpha v_n + q_n) \text{ in } X_c,\\
&v_{n+1} \equiv 0 \text{ in } X\backslash X_c, \quad n = 0, 1, \ldots,
\end{aligned}
\tag{34}
$$

where

$$\tau_2 = 2/(2\alpha + \gamma_2),\ \gamma_2 = 1/(1 - b_1)^2.$$

Similarly to the previous lemma, we have:

Lemma 7 *[4] If $\alpha > 0$ and $\tau = 2/(2\alpha + \gamma_2)$, then algorithm (34) converges, and the following estimate holds:*

$$
\begin{aligned}
&||\phi - \phi_n||_{H_2^1} + ||q - q_n||_{H_2^1} + ||m_c(v - v_n)||_{L_2}\\
&\qquad \leq C \cdot \left(\frac{\gamma_2}{2\alpha + \gamma_2}\right)^n \to 0 \ \textit{as}\ n \to \infty
\end{aligned}
$$

with $C = C(v_0, \phi_{obs}, \phi^{(0)}) = \mathrm{const} > 0$.

Remark. In [6] other classes of iteration methods are formulated, which can be also applied to the problems under consideration. □

4 Description of numerical methods

For simplicity we further denote $\phi_{obs} - \phi^{(0)}$ just by ϕ_{obs}. N is equal to the number of grid points at the axis z, $2M$ is the number of grid points along "μ"-axis. We set $\mu_j = j\Delta\mu$, $-M \leq j \leq M$, $j \neq 0$, $\Delta\mu = 2/(2M - 1)$. Also, $z_i = (i - 1)\Delta z$, $1 \leq i \leq N$, $\Delta z = 1/(N - 1)$. We set $\phi_{i,j} = \phi(\mu_j, z_i)$. For a fixed n (iteration step) problem (31)–(33) is approximated by a finite-difference scheme justified in [1], that can be written in the following form for direct equation (31):

$$
\begin{cases}
\mu_j \dfrac{\phi_{i+1,j} - \phi_{i,j}}{\Delta z} + \phi_{i+1,j} - \dfrac{b}{2} \displaystyle\sum_{\substack{k=-M \\ k \neq 0}}^{k=M} \Delta\mu \, \phi_{i+1,k} = v_{i,j}, \quad i = 1, \ldots, N-1, \ \ j = 1, \ldots, M, \\
\phi_{1,j} = 0, \ \ j = 1, \ldots, M, \\
\mu_j \dfrac{\phi_{i+1,j} - \phi_{i,j}}{\Delta z} + \phi_{i,j} - \dfrac{b}{2} \sum_{\substack{k=-M \\ k \neq 0}}^{k=M} \Delta\mu \, \phi_{i,k} = v_{i,j}, \quad i = 1, \ldots, N-1, \ \ j = -1, \ldots, -M, \\
\phi_{N,j} = 0, \ \ j = -1, \ldots, -M.
\end{cases}
$$

Then we solve adjoint equation (32):

$$
\begin{cases}
-\mu_j \dfrac{q_{i+1,j} - q_{i,j}}{\Delta z} + q_{i+1,j} - \dfrac{b}{2} \sum_{\substack{k=-M \\ k \neq 0}}^{k=M} \Delta\mu \, q_{i+1,k} = 0, \ i = 1, \ldots, N-1, \ j = -1, \ldots, -M, \\
q_{1,j} = (\phi_n - \phi_{obs})(0, \mu_j) \quad j = -1, \ldots, -M, \\
-\mu_j \dfrac{q_{i+1,j} - q_{i,j}}{\Delta z} + q_{i,j} - \dfrac{b}{2} \sum_{\substack{k=-M \\ k \neq 0}}^{k=M} \Delta\mu \, q_{i,k} = 0, \ \ i = 1, \ldots, N-1, \ j = 1, \ldots, M, \\
q_{N,j} = (\phi_n - \phi_{obs})(H, \mu_j), \quad j = 1, \ldots, M.
\end{cases}
$$

To solve the system of arizing systems of linear equations we use method [7]. So, we find ϕ_n and q_n. After that we solve control equation (33), which is approximated as follows:

$$
\begin{aligned}
(v_{n+1})_{i,j} = (v_n)_{i,j}(1 - \alpha\tau_1) - \tau_1 (q_n)_{i,j}, \\
(v_{n+1})_{i,j} = 0, \ (\mu_j, z_i) \in X \setminus X_c.
\end{aligned}
$$

We are ready then to pass to the next, $n+1$-th, iteration step.

Similarly, the computer version of algorithm (34) for Problem 2 is approximated by the following finite-difference scheme:

$$
\begin{cases}
\mu_j \dfrac{\phi_{i+1,j} - \phi_{i,j}}{\Delta z} + \phi_{i+1,j} - \dfrac{b}{2} \displaystyle\sum_{\substack{k=-M \\ k \neq 0}}^{k=M} \Delta\mu\, \phi_{i+1,k} = v_{i,j}, \quad i = 1, \ldots, N-1, \ \ j = 1, \ldots, M, \\
\phi_{1,j} = 0, \ \ j = 1, \ldots, M, \\
\mu_j \dfrac{\phi_{i+1,j} - \phi_{i,j}}{\Delta z} + \phi_{i,j} - \dfrac{b}{2} \sum_{\substack{k=-M \\ k \neq 0}}^{k=M} \Delta\mu\, \phi_{i,k} = v_{i,j}, \quad i = 1, \ldots, N-1, \ \ j = -1, \ldots, -M, \\
\phi_{N,j} = 0, \ \ j = -1, \ldots, -M.
\end{cases}
$$

Then we solve the adjoint equation:

$$
\begin{cases}
-\mu_j \dfrac{q_{i+1,j} - q_{i,j}}{\Delta z} + q_{i+1,j} - \dfrac{b}{2} \sum_{\substack{k=-M \\ k \neq 0}}^{k=M} \Delta\mu\, q_{i+1,k} = (\phi_n - \phi_{obs})_{i,j}, \\
i = 1, \ldots, N-1, \ \ j = -1, \ldots, -M, \\
q_{1,j} = 0, \quad j = -1, \ldots, -M, \\
-\mu_j \dfrac{q_{i+1,j} - q_{i,j}}{\Delta z} + q_{i,j} - \dfrac{b}{2} \sum_{\substack{k=-M \\ k \neq 0}}^{k=M} \Delta\mu\, q_{i,k} = (\phi_n - \phi_{obs})_{i,j}, \\
i = 1, \ldots, N-1, \ \ j = 1, \ldots, M, \\
q_{N,j} = 0, \quad j = 1, \ldots, M.
\end{cases}
$$

After that we solve control equation:

$$
(v_{n+1})_{i,j} = (v_n)_{i,j}(1 - \alpha\tau_2) - \tau_2 (q_n)_{i,j},
$$
$$
(v_{n+1})_{i,j} = 0, \ (\mu_j, z_i) \in X \setminus X_c.
$$

Then we pass to the next iteration step.

As a criterion for finishing iterational processes (21)–(23), (28)–(30), the following inequality is used:

$$
r = \frac{\|v_{n+1} - v_n\|_{L_2}}{\|v_n\|_{L_2}} \leq \varepsilon
$$

with $\varepsilon = 0.0005$. Norm $\|.\|$ is a discrete case of norm $L_2(X)$.

The numerical experiment is performed in accordance with the following scheme:

1. We solve the direct equation with given right-hand side $\hat{v}$ (test source function) and obtain observation data ϕ_{obs}.

2. Using the iterational processes (21)–(23), (28)–(30) (subject to Problems 1, 2) and observation data ϕ_{obs} found at the previous step, we obtain an approximate source solution v.

3. After completing the iterational process we compare the exact test solution $\hat{v}$ and its recovered analogue v. For this case we introduce the residual R that shows the deviation of the recovered solution v from the exact solution $\hat{v}$:

$$R = \frac{\|\hat{v} - v\|_{L_2}}{\|\hat{v}\|_{L_2}}.$$

5 Experimental results

To perform numerical experiments we always set $b(z) \equiv b = 0.5$, $p(\mu, \mu_0) \equiv 1$, $N = 2M$, $\Gamma_+^{(obs)} \equiv \Gamma_+$, and $H = 1$. Hence, $\gamma_1 = 0.925$, $\gamma_2 = 4$, and for $\alpha \leq 0.01$ we approximately have $\tau_1 = 2.115$, $\tau_2 = 0.497$.

Numerical experiment 1. In this case we have reconstructed some test source functions if $X_c = X_{obs} = X$ on the basis of internal observations (Problem 2) using algorithm (34). The value of α is taken equal to $\alpha = 0.01$ and $\tau_2 \approx 0.5$.

First, we treat a test source function $\hat{v} = z(H - z)(1 - \mu^2)$. Its graph and its reconstructed image are presented at Figures 1.1 and 1.2, respectively. We have here the number of iterations $n = 89$ and the residual $R = 0.025$.

Secondly, we consider a test source functions $\hat{v} = \mu z$. Its exact graph and a reconstructed image can be found in Figures 1.3 and 1.4, respectively. Here $n = 95$ and $R = 0.029$.

Thirdly, we consider $\hat{v} = \sin^2(2\pi z)\cos(\frac{1}{2}\pi\mu)$. The number of grid points is equal to 26 ($N = 2M = 26$). Figures 1.5 and 1.6 show the exact source function and its recovered analogue, respectively. To obtain better result we have taken at this point $\alpha = 0.005$, and we have here $R = 0.06$, $n = 186$.

Figure 1.1. Exact solution $\hat{v} = z(H - z)(1 - \mu^2)$.

Figure 1.2. Numerical solution v,
$X_c = X_{obs} = X,\ \alpha = 0.01,\ n = 89,\ R = 0.025.$

Figure 1.3. Exact solution $\hat{v} = \mu z,\ H = 1.$

Figure 1.4. Numerical solution v,
$X_c = X_{obs} = X,\ \alpha = 0.01,\ n = 95,\ R = 0.029.$

Figure 1.5. Exact solution $\hat{v} = \sin^2(2\pi z)\cos(\frac{1}{2}\pi\mu),\ H = 1.$

Figure 1.6. Numerical solution v,
$X_c = X_{obs} = X,\ \alpha = 0.005,\ n = 186,\ R = 0.06,\ N = 26.$

Numerical experiment 2.

In this case we reconstruct an isotropic (μ-independent) source function on the basis of boundary observations using algorithm (31), (32), and the following correlation

$$v_{n+1} = v_n - \tau_1\Big(\alpha m_c v_n + \frac{1}{2}\int_{-1}^{1} m_c q_n d\mu\Big).$$

This replacement of (33) is caused by (24).

As a test we take source function $\hat{v} = 1$ on $X_c = [-1, 1] \times [0.3H, 0.7H]$ and zero otherwise. As usually, $H = 1$.

Figure 2.1 shows the exact value of $\hat{v}$, Figure 2.2 describes restored function v if $X_{obs} = \Gamma_+$ (Problem 1), and, just to compare, Figure 2.3 shows restored function v if $X_{obs} = X \supset X_c$ (Problem 2). The result is obtained at the same $\alpha = 0.01$.

It is seen explicitely that the function v is well reconstructed in both cases and there is no need to use the overdetermined information given on $X_{obs} = X$ because the result obtained from the boundary observation is rather satisfactory and needs less iterations.

Figure 2.1. Exact solution $\hat{v} = 1,\ X_c = [0.3H, 0.7H],\ H = 1.$

Figure 2.2. Numerical solution v for $X_{obs} = \Gamma_+$.
$n = 14,\ \alpha = 0.01,\ R = 0.0446.$

Figure 2.3. Numerical solution v for $X_{obs} = X \supset X_c$.
$n = 96,\ \alpha = 0.01,\ R = 0.053.$

Numerical experiment 3.

In this experiments we reconstruct source function in the case of unsufficient internal information: $X_c \not\subset X_{obs}$. So the conditions of Lemma 5 do not hold. We use algorithm (34) with control equation in the most general form (30) and set $\hat{v} = 1$ in X_c.

First, X_c has a nonzero intersection with X_{obs}. We set $X_c = [0.07, 0.55]$, $X_{obs} = [0.3, 1]$, $\mu \in [-1, 1]$. Exact and reconstructed source functions are presented in Figures 3.1 and 3.2, respectively.

If $X_c \cap X_{obs} = \emptyset$ then the result is worse, it is demonstrated in Figure 3.3 for $X_{obs} = [0.6, 1]$.

The last case, $X_{obs} \subset X_c$, is considered for $X_{obs} = [0.3, 0.45]$. The result is very bad, too (Figure 3.4).

So, these numerical results demonstrate that internal reconstructing on the basis of lack of information cannot, generally speaking, give us a suitable result.

Figure 3.1. Exact solution $\hat{v} = 1$, $X_c = [0.07, 0.55]$.

Figure 3.2. Numerical solution. $X_c \cap X_{obs} \neq \emptyset$,
$X_{obs} = [0.3, 1],\ R = 0.48,\ n = 130,\ \alpha = 0.01.$

Figure 3.3. Numerical solution. $X_c \cap X_{obs} = \emptyset$,
$X_{obs} = [0.6, 1]$, $R = 0.663$, $n = 74$, $\alpha = 0.01$.

Figure 3.4. $X_{obs} \subset X_c,\ X_{obs} = [0.3, 0.45],\ R = 0.549,\ n = 133,\ \alpha = 0.01.$

Numerical experiment 4.

In these experiments we use boundary observations $X_{obs} = \Gamma_+$ for reconstructing source function in space $L_2(X)$. We do not assume that $v \in U_i$, $i = 1, 2, \ldots, 5$. The conditions of Lemmas 1–3 do not hold. Therefore there exists a nonzero kernel and we cannot use an improved form of control equation (e.g., (24)–(27)). So, we use control equation in the general form (34). In this case we also cannot expect a correct reconstruction. Our numerical results demonstrate these reasonings. We take a test solution $\phi(\mu, z) = z(H - z)$, then find the corresponding source function $\hat{v}(\mu, z) = \mu(H - 2z) + (1 - b)z(H - z)$ (Figure 4.1). Obviously, this sourse function belongs to the kernel of operator B. After that we take the trace $\phi|_{\Gamma_+}$ and reconstruct the function v (Figure 4.2). To minimize functional J_1 this result goes to zero (but not to the desired $\hat{v}$!). In Figure 4.3 we demonstrate this fact by drawing the graphs of $\hat{v}$ and v at $z = 0.5H$ $(H = 1)$.

Figure 4.1. Exact solution $\hat{v} = \mu(H - 2z) + (1 - b)z(H - z)$.

Figure 4.2. Numerical solution v. $X_{obs} = \Gamma_+$. Nonzero kernel. Incorrect reconstruction ($\alpha = 0.01$).

Figure 4.3. $X_{obs} = \Gamma_+$. Correct function $\hat{v}$ and incorrect reconstructing at $z = 0.5H$.

Numerical experiment 5. For the case of Problem 2 (internal observations) with test function $\hat{v} = 0.25z(H-z)$ and $X_c = X_{obs} = X$, we present here the dependence of residual R, number of iterations n and value of functional J_2 on the parameter α (Figures 5.1, 5.2, 5.3, respectively). Besides Figures, these results are presented in Table 1 below.

Table 1.

α	1.5	1	0.5	0.1	0.05	0.01	0.001
R	0.669	0.594	0.467	0.234	0.163	0.059	0.0186
n	13	14	16	36	50	94	148
$J_2 \cdot 10^5$	332	271	177	49.7	26.4	5.02	0.548

The results in Table 2 demonstrate the dependence of the residual and the number of iterations on the number of grid points $N = 2M$. Here $\alpha = 0.01$, the test function is taken equal to $\hat{v} = 0.25z(H-z)$. We can see that in such simple cases there is no need to increase the number of grid points.

Table 2.

$N = 2M$	10	16	20	26
R	0.0542	0.0582	0.0593	0.0604
n	92	93	94	94

Figure 5.1. $R = R(\alpha)$, $X_c = X_{obs} = X$.

Figure 5.2. $n = n(\alpha)$, $X_c = X_{obs} = X$.

Figure 5.3. $J_2 = J_2(\alpha)$, $X_c = X_{obs} = X$.

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