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Existence Theorem for Spatially Inhomogeneous Boltzmann Equation from the Coagulation-Fragmentation Dynamics [1]

One of the most important mechanisms of evolution of disperse systems consisting of two phases one of which is distributed into another one, is the mechanism of coagulation (merging) and fragmentation of particles. This process in the spatially inhomogeneous case may be modelled by the following kind of Boltzmann equation [7]:

$$\frac{\partial c(x,z,t)}{\partial t}+\mathrm{div}_z(v(x,z,t)c(x,z,t))=\frac{1}{2}\int_0^x K(x-y,y)c(x-y,z,t)c(y,z,t)dy-$$

$$-c(x,z,t)\int_0^\infty K(x,y)c(y,z,t)dy+q(x,z,t)-a(x,z,t)c(x,z,t)+$$

$$+\int_x^\infty F(y-x,x)c(y,z,t)-\frac{1}{2}c(x,z,t)\int_0^x F(x-y,y)dy,\quad x\geq 0, t>0, \tag{1}$$

$$c(x,z,0)=c_0(x,z)\geq 0,\quad x\geq 0. \tag{2}$$

Here $c(x,z,t)$ is the distribution function of particles with mass $x\in R^1_+$ at the time moment $t\geq 0$ in the space point $z=(z_1,z_2,z_3)\in R^3$. Nonnegative symmetric functions K ΥF called the coagulation and fragmentation kernels are supposed to be known from the physical background of the process. Their influence on the particle evolution is the most essential because they characterize the intensity of merging of particles of masses x and y and splitting of particles with the mass $x+y$ onto two particles with masses x and y. From physical point of view these kernels should be symmetric, i.e. $K(x,y)=K(y,x)$, $F(x,y)=F(y,x)$ for all arguments. The function $v(x,z,t)\in R^3$ is the velocity of spatial transfer of particles of mass x in the point z at the time t. The scalar function $q(x,z,t)$ is the distribution function of external sources of particles, and $a(x,z,t)$ defines the intensity of absorbtion of particles by external medium, i.e. it is proportional to effluxes. The functions v,q,a are supposed to be known.

The initial value problem (1),(2) without fragmentation ($F\equiv 0$) was studied in [1,5,6,3] where a number of existence and uniqueness theorems was established on different assumptions on the coagulation kernels, space velocity and initial data.

In this paper we prove existence of global continuous solution of the initial value problem (1), (2).

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First, we fix some $T > 0$ and write the problem (1),(2) in the following integral form

$$c(x,z,t) = c_0(x,z^{(0)}) + \int_0^t \left(\mathbf{S}(c) - c\,\mathrm{div}_z v\right)(x, z(s), s)\,ds. \tag{3}$$

Here $z^{(0)}$ is the starting point of the characteristic of equation (1) passing through the point (x,z,t); the collision operator $\mathbf{S}$ is equal to the right-hand side of the equation (1). We assume that the characteristic equation has a unique solution $z(x,s)$.

We impose the following conditions on space transfer and effluxes which do not restrict the variety of physically meaningful processes[2]:

$$a(x,z,t) + \mathrm{div}_z v(x,z,t) \geq -b = \mathrm{const},\ 0 \leq t \leq T, \quad z \in R^3_+, \quad x \geq 0. \tag{4}$$

The functions a, v Ƴ$\mathrm{div}_z v$ are assumed to be continuous.

Let for some $b(x) \geq 0$ we have the following apriori estimate of the solution to (1), (2):

$$\sup_{0 \leq t \leq T} \sup_{z \in R^3} \int_0^\infty b(x) c(x,z,t)\,dx \leq M = \mathrm{const} < \infty. \tag{5}$$

This estimate can be proved in some cases [4]. If $b \equiv$ const and the coagulation kernel is bounded along with decreasing fragmentation one then (5) allows to prove existence of a generalized solution. We show now that sometimes the estimate (5) may be improved by moving the supremum over z into the integral. This minor correction yields drastically different result allowing to prove existence of a continuous solution.

Lemma 1 *Let (5) hold and there exist a monotonically increasing nonnegative function $\phi(x)$ such that*

$$\int_0^\infty \frac{dx}{\phi(x)} < \infty.$$

Let

$$\sup_{x,z} \phi(x) c_0(x,z) = M_0 < \infty, \tag{6}$$

$$\int_0^T \sup_{x \geq 0, z \in R^3} \phi(x) q(x,z,s)\,ds = Q < \infty; \tag{7}$$

and

$$\phi(x) K(x-y,y) \leq b(y) \cdot \phi(x-y).\ y \in [0, x/2] \tag{8}$$

Assume that at least one of two following conditions on the fragmentation

$$\int_x^\infty \frac{F(y-x,x)}{\phi(y)}\,dy \leq \frac{r}{\phi(x)}, \quad x > 0,\ r = const \tag{9}$$

or

$$\phi(x) F(y-x,x) \leq b(y) \tag{10}$$

is valid. Then the estimate (5) can be improved:

$$\|c\| = \sup_{0 \leq t \leq T} \int_0^\infty \sup_z |c(x,z,t)|\,dx \leq const. \tag{11}$$

[2]In fact, from the continuity equation we see $\partial\rho/\partial t + \mathrm{div}_z(\rho v) = 0$ and from boundness of the density ρ and its derivatives we have (4). This observation was pointed out to the author by A.E.Aloyan.

Proof. By throwing away the negative terms in (3) we obtain the inequality

$$c^n(x,z,t) \le \sup_z c_0(x,z) + \frac{1}{2}\int_{t_0}^{t}\int_0^x K(x-y,y)c^n(x-y,z(x,s),s)c^n(y,z(x,s),s)dyds$$

$$+\int_{t_0}^{t}\left\{\int_x^\infty F(y-x,x)c^n(y,z(x,s),s)dy + q(x,z(x,s),s)\right\}ds. \tag{12}$$

Multiplying (12) by $\phi(x)$ we have with (8) taken into account

$$\phi(x)c^n(x,z,t) \le M_0 + \int_0^t \left[\sup_{x,z}[\phi(x)c(x,z,s)]\int_0^{x/2} b(y)c(y,z(x,s),s)dy\right.$$

$$\left. + \phi(x)\int_x^\infty F(y-x,x)c(y,z(x,s),s)dy + \phi(x)q(x,z(x,s),s)\right]ds. \tag{13}$$

We introduce the notation

$$\sup_{x,z}\phi(x)c(x,z,t) = f(t) \tag{14}$$

and extract $f(t)$ from the second integral summand in (13):

$$\phi(x)\int_x^\infty F(y-x,x)c(y,z(x,s),s)dy \le \begin{cases} rf(s) & \text{for } (9); \\ \int_0^\infty b(y)c(y,z(x,s),s)dy & \text{for } (10). \end{cases}$$

By summarizing both cases we strengthen the inequality and obtain

$$f(t) \le M_1 + \int_0^t (f(s)(1+r)+1)\left[\int_0^\infty b(y)c(y,z,s)dy\right]ds.$$

Here $M_1 = M_0 + Q$. Hence, taking into account (5) we come to the correlation

$$f(t) \le (M_1+M)\mathrm{e}^{M(1+r)t} \le M_2, \quad 0 \le t \le T. \tag{15}$$

From (14), (15) we establish

$$c(x,z,t) \le M_2/\phi(x), \quad x \ge 0, \quad z \in R^3, \quad t \in [0,T].$$

Consequently,

$$\|c\| \le M_2\int_0^\infty \frac{dx}{\phi(x)} = \text{const}.$$

This proves Lemma 1.

Remark. The conditions of Lemma 1 are satisfied, e.g., for

$$K(x,y) \le \text{const}\cdot\exp(-\alpha(x+y)), \quad F(x,y) \le \text{const}, \quad \phi(x) = \exp(\alpha x),\ \alpha > 0;$$

$$K(x,y) \le \text{const}, \quad F(x,y) \le \text{const}\cdot(1+x+y)^{-1}, \quad \phi(x) = (1+x)^k,\ k > 1.$$

We are now in position to formulate the global existence theorem for (1), (2).

Theorem 1 *Let the conditions of Lemma 1 be valid. Let for sufficiently large A the integral*

$$\int_A^\infty \frac{K(x,y)}{\phi(y)}dy, \qquad x \in [0,B]$$

be bounded for any $B < \infty$ uniformly in x. Then there exists at least one continuous nonnegative solution to the initial value problem (1), (2).

The proof is based on introducing the sequence $\{c^n\}_{n=1}^\infty$ of solutions of approximated problems with truncated compactly supported integral kernels. These kernels should tend in the space C to the original ones on each compacta. Similarly to [2, Lemma 2.2, Step 1] we show that $\{c^n\}_{n=1}^\infty$ is uniformly bounded on each compacta. The estimate (11) is uniform one with regard to this sequence. Hence, it is possible to pass to the limit under integral. Therefore we choose from $\{c^n\}_{n=1}^\infty$ a subsequence converging in C on a compact set. Then we double this compact set and choose a subsubsequence and so on. Finally, after this standard diagonal process, we have a subsequence tending to a continuous function $c(x,z,t)$ on each compact set in $[0,\infty) \times R^3 \times [0,T]$. Due to arbitrariness of T we get global existence of a continuous solution.

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