Comment on singular solutions to the stationary coagulation equation

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Abstract

We comment on our paper "Exact solutions for the coagulation–fragmentation equation".

We examine the stationary Smoluchowski coagulation equation

$$\frac{1}{2}\int_0^m K(m-m_1,m_1)c(m-m_1)c(m_1)dm_1 - c(m)\int_0^\infty K(m,m_1)c(m_1)dm_1 = 0 \quad (1)$$

with a symmetric non–negative coagulation kernel $K(m,m_1) = K(m_1,m)$, $m,m_1 \geq 0$. A previous paper by the authors [1] demonstrated the surprising phenomenon of existence of stationary solutions to (1). A typical solution with this behaviour satisfies the equality

$$K(m,m_1)c(m)c(m_1) = \frac{1}{(m+m_1)^3}. \quad (2)$$

The bounded continuous coagulation kernel

$$K(m,m_1) = \frac{v(m)v(m_1)}{(m+m_1)^3} \quad \text{where} \quad v(m) = \begin{cases} m^3, & 0 \leq m \leq 1 \\ 1, & m \geq 1 \end{cases} \quad (3)$$

yields the stationary solution

$$c(m) = \begin{cases} m^{-3}, & 0 < m \leq 1 \\ 1, & m \geq 1 \end{cases}. \quad (4)$$

This mathematical phenomenon is really surprising and an attempt should be made to discuss this result from a physical point of view.

By considering the Smoluchowski model in the form

$$\frac{\partial}{\partial t}c(m,t) = \frac{1}{2}\int_0^m K(m-m_1,m_1)c(m-m_1,t)c(m_1,t)dm_1 -$$

$$-c(m,t)\int_0^\infty K(m,m_1)c(m_1,t)dm_1, \quad m>0 \tag{5}$$

$$\frac{\partial}{\partial t}c(0,t) = -c(0,t)\int_0^\infty K(0,m_1)c(m_1,t)dm_1, \quad m=0. \tag{6}$$

an attempt has been made in [2] to explain the occurrence of the above stationary solution. A similar approach connected with replacing Smoluchowski's model by another one was also considered by others when it was discovered that the coagulation kernel $K(m,m_1) = mm_1$ yields the paradoxical infringement of the mass conservation law after a critical time (see [3], [4] and refs. in [1]). There were suggestions to change the Smoluchowski equation to ensure the conservation of mass by replacing the second term in (1) with

$$mc(m,t)\int_0^\infty m_1 c(m_1,0)dm_1$$

(see e.g. discussion in [3]). However, further research reported in the literature demonstrated the correctness of the original coagulation model [3,4]. This adapted coagulation model did not replace the original Smoluchowski equation. We anticipate that the model (5), (6) is destined for a similar fate: the reason is that in [2] it is assumed that the value of the function

$$f(m) = \int_0^m G(m,m_1)dm_1 \tag{7}$$

at the point $m=0$ is equal to zero independently of the integrand $G(m,m_1)$. However, an elementary computation can show that, for instance, the value of the function

$$f(m) = \int_0^m \frac{1}{m+m_1}dm_1 \tag{8}$$

at the point $m = 0$ is equal to $\ln 2$ despite the integrand being singular at zero. Following the argument presented in [2] it would have to be equal to zero which, as demonstrated by (8), cannot be true in general. A similar problem arises in the well known solution to the heat equation

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty} \mathrm{e}^{-(x-y)^2/4t}u_0(y)dy \tag{9}$$

at the initial time $t = 0$: direct substitution of $t = 0$ into (9) is impossible and hence we must pass to the limit $t \to 0$ in order to define $u(x, 0)$ from (9).

The simple examples (8) and (9) show that the particular approach in [2] is fundamentally different to that in [1]. In [1] we study the Smoluchowski equation from the mathematical point of view, the value at zero being considered as the limit as $m \to 0$, all the results being mathematically rigorous.

The above discussion demonstrates that the model (5), (6) leads us away from a physical explanation of the mathematical phenomenon discovered in [1]. As an attempt to explain the phenomenon let us consider the bounded coagulation kernel (3) with the stationary solution (4). We can see that this stationary state is achieved due to a constant influx of small particles. The number of small particles is infinite and, moreover, is large enough to ensure the influx of small particles for all $t > 0$. The phenomenon of existence of stationary solutions of the pure coagulation equation therefore takes place due to the influence of a sufficiently large "infinity" of small particles. In conclusion, we would like to add that the above mentioned phenomenon of the infringement of the mass conservation law for $K(m, m_1) = mm_1$ is due to a different type of influence of infinity.

References

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