Existence and uniqueness theorem for the coagulation equation with space inhomogeneous velocity fields

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Abstract

We prove the global existence and uniqueness theorem to the space nonhomogeneous Smoluchovsky coagulation equation for unbounded coagulation kernels with multiplicative growth on infinity . These kernels include almost all physically reasonable cases. Initial data are supposed to be sufficiently small.

We examine the asymptotic properties of the general coagulation equation which for space inhomogeneous case can be written as

$$\frac{\partial c(x,z,t)}{\partial t} + div_z(v(x,z)c(x,z,t)) =$$
$$= \frac{1}{2}\int_0^x K(x-y,y)c(x-y,z,t)c(y,z,t)dy -$$
$$-c(x,z,t)\int_0^\infty K(x,y)c(y,z,t)dy, \tag{1}$$

$$c(x,z,0) = c_0(x,z) \geq 0. \tag{2}$$

It describes the time evolution of particles in disperse systems with distribution function $c(x,z,t) \geq 0$ of mass $x \geq 0$ at time $t \geq 0$ at space point $z \in R^3$ whose change in mass governed by the non-negative reaction rate K which is called the coagulation kernel. The coagulation kernel K models the rate

at which particles of mass x coalesce with those of mass y. Applications of (1) can be found in many problems including chemistry (e.g. reacting polymers), physics (aggregation of colloidal particles, growth of gas bubbles in solids), astrophysics (formation of stars and planets), meteorology (merging of drops in atmospheric clouds).
This spatially inhomogeneous equation was studied, e.g., in [1, 2] for bounded kernels K, in [3] for unbounded coagulation kernels of special type. In this note we are concerned with kernels $K(x,y) \leq k(1+x)(1+y)$ which include the main part of physically reasonable cases.

We assume that there exists $\delta > 0$ such that

$$\mathrm{div}_z \; v(x,z) \geq \delta, \qquad x \in [0,\infty), \quad z \in R^3. \tag{3}$$

This condition holds, e.g., for particles moving in gravitation field. Particularly, for a particle with mass x and zero initial velocity, falling on a star with mass X from a distance z_0, we have

$$\inf_z \frac{\partial v(x,z)}{\partial z} = \frac{16\sqrt{3}}{9\sqrt{2}} (\gamma X)^{\frac{1}{2}} z_0^{-3/2}$$

where γ is the gravitational constant.
We introduce the space Ω_λ of continuous functions with a bounded norm

$$\|c\|_\lambda = \sup_{0 \leq t < \infty} \int_0^\infty \exp(\lambda x) \sup_{z \in R^3} |c(x,z,t)| dx.$$

We shall use the following notations:

$$\Omega = \bigcup_{\lambda > 0} \Omega_\lambda, \; \Omega^+ = \{c : \; c \geq 0, \; c \in \Omega\}.$$

In this paper we prove the following theorem.

Theorem 1 *Let nonnegative continuous coagulation kernel K satisfies the inequality*

$$K(x,y) \leq k(1+x)(1+y), \quad (x,y) \in R_+^2, k = const. \tag{4}$$

Let $v \in C^1$ and (3) holds. Suppose $c_0 \in \Omega^+$ and

$$c_0(x,z) \leq A \exp(-ax), \quad x \in R_+^1, \quad z \in R^3, \quad a > 0. \tag{5}$$

Then there exist positive constants A and a such that the problem (1), (2) has global in time solution $c \in \Omega^+$. This solution is unique in class Ω if the following additional condition provided:

$$div_z\ v \le \Delta = const, \quad x \in [0,\infty), \quad z \in R^3.$$

Lemma 1 *Let conditions of Theorem hold and the coagulation kernel has a compact support. Then there exists at least one solution to the problem (1), (2) in class Ω^+.*

Local existence is proved using the contraction mapping theorem. Nonnegativeness of solution has been proved for more general case in lemma 2 in [5]. Extension of solution for all $t > 0$ bases on apriori estimation of boundedness on each compact, which was established in [6].

Lemma 2 *Let conditions of Theorem hold. Then there exist positive constants λ, C_0 such that any continuous nonnegative solution to (1),(2) obeys the inequality*

$$c(x,z,t) \le C_0(1+x)^{-1}\exp(-\lambda x)\exp(-\delta t). \tag{6}$$

Proof. We use the substitution

$$c(x,z,t) = g(x,z,\tau)\exp(-\delta t)(1+x)^{-1}, \quad \tau = 1-\exp(-\delta t),\ \tau \in [0,1). \tag{7}$$

Then from (1),(2) we obtain

$$\delta\frac{\partial g}{\partial \tau} + (1-\tau)^{-1}(v(x,z),\nabla_z g) =$$
$$\frac{1}{2}(1+x)\int_0^x \frac{K(x-y,y)}{(1+x-y)(1+y)}g(x-y)g(y)dy$$
$$-g(x)\int_0^\infty \frac{K(x,y)}{1+y}g(y)dy - [\text{div }v - \delta](1-\tau)^{-1}g(x), \tag{8}$$
$$g|_{\tau=0} = (1+x)c_0(x,z) = g_0(x,z). \tag{9}$$

We write (8), (9) in the integral form and take $\sup_{z\in R^3}$. Then we have $\sup_{z\in R^3} g(x,z,t) \le f(x,t)$ where f is the right-hand side of the inequality obtained. Finally, we establish with (3),(4),(5) taken into account, that the majorant nonnegative function f satisfies the following inequality

$$\delta\frac{\partial f}{\partial \tau} \le \frac{1}{2}k(1+x)f * f(x), \tag{10}$$

$$f|_{\tau=0} = f_0(x) < A(1+x)\exp(-ax) \le A_\varepsilon \exp(-(a-\varepsilon)x) \tag{11}$$

where $f * f$ means the convolution: $f * f(x) = \int_0^x f(x-y)f(y)dy$ and

$$A_\varepsilon = \varepsilon^{-1}\exp(\varepsilon - 1), \quad 0 < \varepsilon < \min\{a, 1\}. \tag{12}$$

For $x \le 1$ we have

$$\frac{\partial f}{\partial \tau} \le k\delta^{-1}\ f * f, \tag{13}$$

$$f_0(x) < 2A\exp(-ax). \tag{14}$$

Changing the inequalities (13),(14) to equalities and solving the equation obtained, we can see that

$$f(x,\tau)|_{x\le 1} \le 2a\exp((2\delta^{-1}Ak\tau - a)x).$$

Consequently, if $0 \le x, \tau \le 1$, then

$$f(x,\tau) \le \hat{A} = \begin{cases} 2A, & A \le a\delta/(2k), \\ 2A\exp(2Ak\delta^{-1} - a), & A > a\delta/(2k). \end{cases} \tag{15}$$

We are in position now to prove that $f < h$ if the "upper" function h satisfies the following equation:

$$\frac{\partial h}{\partial \tau} = k\delta^{-1}xh * h(x), \tag{16}$$

$$h|_{\tau=0} = h_0(x) = C_0\exp(-(a-\varepsilon)x) \tag{17}$$

where

$$C_0 > \max\{A_\varepsilon,\ \hat{A}\exp(a-\varepsilon)\}. \tag{18}$$

Expressions (10), (15), (17), (18) ensure

$$h_0(x) > f_0(x),\ x \ge 0 \ \text{ and } \ h_0(x) > f(x,\tau),\ 0 \le x, \tau \le 1. \tag{19}$$

Using the Laplace transform we can solve (16),(17) and obtain

$$h(x,\tau) = \sum_{n=1}^{\infty} \left(4kx^2\tau\delta^{-1}\right)^{n-1} C_0^n \frac{(2n-3)!!}{n!(2n-2)!}\exp(-(a-\varepsilon)x) \tag{20}$$

with $(2n-3)!! = 1 \cdot 3 \cdot 5 \cdot ... \cdot (2n-3)$, $(-1)!! = 1$. Using the equality $(2n)!! = 2^n n!$, we establish the inequality

$$h(x,\tau) \le C_0\cosh\left(\sqrt{8kC_0\delta^{-1}}x\right)\exp(-(a-\varepsilon)x),\ \tau \in [0,1),\ x \ge 0. \tag{21}$$

¿From (20) we see that the function h is continuous and increases in τ. Consequently, $h(x,\tau) > f(x,\tau)$ for $0 \le x, \tau < 1$. To demonstrate $f < h$ we assume that there is a set D of points (x,τ) on which $f(x,\tau) = h(x,\tau)$. It is worth to note that (19) yields $x > 1, \tau > 0$ if $(x,\tau) \in D$. We choose $(x_0, \tau_0) \in D$ so that the rectangle $Q = [0, x_0) \times [0, \tau_0)$ contains no points of D. As far h and f are continuous, we have $f(x,\tau) < h(x,\tau)$ for $(x,\tau) \in Q$. Hence

$$f(x_0,\tau_0) = h(x_0,\tau_0) = h_0(x_0) + kx_0\delta^{-1}\int_0^{\tau_0} h * h(x_0,s)ds >$$

$$> f_0(x_0) + \frac{1}{2}k\delta^{-1}(1+x_0)\int_0^{\tau_0} f * f(x_0,s)ds \ge f(x_0,\tau_0).$$

We arrive at the contradiction $f(x_0,\tau_0) > f(x_0,\tau_0)$, which proves that D is empty and $f(x,\tau) < h(x,\tau), x \ge 0,\ 0 \le \tau < 1$. ¿From (21) we see that if we impose the condition

$$\sqrt{8kC_0\delta^{-1}} < a \tag{22}$$

then the assertion of lemma 2 holds with $\lambda = a - \varepsilon - \sqrt{8kC_0\delta^{-1}} > 0$.

Proof of the Theorem. We construct the sequence of continuous symmetric nonnegative kernels $\{K_n\}, n \ge 1$ with compact supports such that $K_n(x,y) = K(x,y)$ if $x, y \le n$. In accordance with assertion of Lemma 1 this sequence borns the sequence $c_n \in \Omega^+$ of nonnegative solutions of (1),(2). Due to (7), (21) we have the important estimation

$$c_n(x,z,t) < \frac{C_0}{1+x}\exp\Big((\sqrt{8kC_0\delta^{-1}} - a + \varepsilon)x - \delta t\Big). \tag{23}$$

Using approach from [6], we show that the sequence $\{c_n\}$ is relatively compact in space of continious functions with the topology of uniform convergence on each compact. By standard diagonal process we build a continuous nonnegative function c such that $c_n \to c$ on each compact. Passing to the limit is possible due to the estimation (23) which ensures uniform smallness of integral "tails" $\int_m^\infty (1+x)c_n(x,z,t)dx$. Therefore the limit function c is a solution to (1), (2). This completes the proof of existence. Uniqueness is proved analogiously to [6]. The Theorem is proved.

Remarks.

1. The suitable constants A and a in (5) can be found from (12), (15), (17), (22).

2. From (6) we see that solution goes to zero as $t \to \infty$.

3. The assertion of Theorem holds if the disperse system concerned has effluence or absorption which correspond to adding the term $\delta c(x)$ to the left-hand side of equation (1).

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