Trend to Equilibrium for the Coagulation-Fragmentation Equation

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Abstract

For a linear coagulation kernel and a constant fragmentation kernel we prove the existence of equilibrium solutions and examine asymptotic properties for time-dependent solutions which are proved to converge to the equilibria. The rate of the convergence is estimated. It is shown also that all time-dependent solutions with the same density can tend to only one particular steady-state solution. In this sense the equilibrium solution is proved to be unique. Existence, uniqueness and mass conservation of time-dependent solutions has been proved in a previous paper by the authors [10].

1 Introduction

We examine the asymptotic properties of the general coagulation-fragmentation equation which can be written as

$$\begin{aligned}\frac{\partial}{\partial t}c(x,t) &= \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy \\ &\quad -c(x,t)\int_0^\infty K(x,y)c(y,t)dy - \frac{1}{2}c(x,t)\int_0^x F(x-y,y)dy\end{aligned}$$

$$+\int_0^\infty F(x,y)c(x+y,t)dy, \tag{1.1}$$

$$c(x,0) = c_0(x) \geq 0. \tag{1.2}$$

Equation (1.1) describes the distribution function of particles $c(x,t) \geq 0$ of mass $x \geq 0$ at time $t \geq 0$ whose change in mass is governed by the non-negative reaction rates K and F which are called, respectively, the coagulation and fragmentation kernels. This paper is a natural continuation of our paper [10]

Carr [6] demonstrated existence of a discrete equilibrium solution and convergence of the time-dependent solution to equilibrium provided that the detailed balance condition for coagulation and fragmentation kernels holds. Mathematically this condition means

$$K(x,y)Q(x)Q(y) = F(x,y)Q(x+y) \tag{1.3}$$

for a positive function Q(x) and immediately gives us the equilibrium solution $\bar{c}(x) = \exp(\lambda x)Q(x)$ in which case there is no necessity to prove its existence. The detailed balance condition leads to the separate cancellation of each pair

$$\frac{1}{2}\int_0^x K(x-y,y)\bar{c}(x-y)\bar{c}(y)dy - \frac{1}{2}\bar{c}(x)\int_0^x F(x-y,y)dy = 0$$

and

$$\int_0^\infty F(x,y)\bar{c}(x+y)dy - \bar{c}(x)\int_0^\infty K(x,y)\bar{c}(y)dy = 0.$$

We do not have this effective cancellation in our work presented below since we do not assume the detailed balance condition. In such circumstances the existence of an equiluilibrium solution is not clear and we therefore remedy this situation in this paper where we discuss the existence and uniqueness of an equilibrium solution. We should observe also that an essential mathematical difference between discrete and continuous models of coagulation-fragmentation consists in the fact that the space l^1 is contained in l^∞ for the discrete case which is not true in the continuous case. Therefore the continuous version (which we treat in this paper) ought to include additional estimates.

Convergence in time of solutions to equilibrium has been studied for the Becker-Döring equations in [2, 3, 13, 14]. These equations coincide with (1.1) in its discrete form when only the interactions of particles with masses 1 and k are permitted.

For the case $K = a, F = b$ with a, b constants the convergence to equilibrium of solutions has been studied via a Lyapunov function by Aizenman and Bak [1] and by the authors [15]. With the aid of Laplace transforms Barrow [4] has considered the large time evolution of solutions for this case, but has not discussed the stability of the resulting equilibria. Equilibrium solutions of the form $\exp(-\lambda x)$ have arisen in both of the above references and are also known to be special cases of more general possible equilibrium solutions [9]. The case of nonexistence of an equilibrium solution was treated by Dubovskiĭ [7]. The asymptotic properties of coagulation models have also been considered by Bruno *et al* [5] and Gajewski [12].

It is our aim to prove existence and uniqueness theorems for equilibrium solutions and examine equilibria which are related to the initial data in the time-dependent problem. We investigate the large time behaviour of solutions near equilibrium for the case $K = a + k(x + y),\ F = b$, where a, b and k are non-negative constants. It is worth pointing out that for this case the hypotheses (H2) and (H4) of [6] do not hold and we therefore utilize a completely different method. The proof of existence of equilibrium in our case is not trivial and is based on three steps. First, we demonstrate that there exists a continuous and positive function which satisfies a special form of (1.1). Second, we prove that time-dependent solutions of (1.1) converge to the function in step one (which at this point is a possible candidate for the equilibrium solution) as time goes to infinity. Third, using properties of time-dependent solutions (which are proved in the second section of this paper) we prove integrability of the function derived in step one and hence establish that it really is the equilibrium solution to (1.1). The existence of equilibrium is also not trivial from the physical point of view. If we apply the above mentioned detailed balance condition (1.3) for $K(x, y) \equiv 1$, $Q(x) = \exp(x^2)$ then we obtain $F(x, y) = \exp(-2xy)$ and $\bar{c}(x) = \exp(x^2 + \lambda x)$. This function is not integrable on $[0, \infty)$ and, consequently, a time-dependent solution of (1.1), (1.2) with bounded mass (expressed by its first moment) cannot converge to such an equilibrium since its total mass must remain constant in accordance with the results of [10].

2 Existence and uniqueness of an equilibrium solution

We are concerned with kernels of the form

$$K(x, y) = a + k(x + y) + g\ xy, \quad F = b \tag{2.1}$$

with non-negative constants $a,\ b,\ k,\ g$. An equilibrium solution $\bar{c}(x)$ to (1.1) has to satisfy the following equation

$$\frac{1}{2}\int_0^x K(x-y,y)\bar{c}(x-y)\bar{c}(y)dy - \bar{c}(x)\int_0^\infty K(x,y)\bar{c}(y)dy$$
$$-\frac{1}{2}\bar{c}(x)\int_0^x F(x-y,y)dy + \int_0^\infty F(x,y)\bar{c}(x+y)dy = 0. \quad (2.2)$$

Let

$$N = \int_0^\infty \bar{c}(x)dx, M = \int_0^\infty x\bar{c}(x)dx.$$

Then, integrating (2.2) and taking into account (2.1), we obtain

$$\frac{a}{2}N^2 + kNM + (gM-b)\frac{M}{2} = 0. \quad (2.3)$$

Therefore

$$N = \frac{1}{a}\sqrt{(k^2-ag)M^2 + abM} - \frac{kM}{a}.$$

From (2.3) we conclude that nonzero non-negative equilibrium solutions cannot exist if $M > b/g$. If $a = k = 0$ then $M = b/g$. If $g = 0$ then an equilibrium solution may be exists for any $M > 0$. Denoting the convolution operator by $*$ and using (2.1) we may rewrite (2.2) in the form

$$\bar{c}(x) = \left(\frac{a}{2}\bar{c}*\bar{c}(x) + \frac{kx}{2}\bar{c}*\bar{c}(x) + \frac{g}{2}(x\bar{c})*(x\bar{c})(x) + bN - \bar{c}*b(x)\right)$$
$$\cdot\frac{1}{aN + kM + x(b/2 + kN + gM)}. \quad (2.4)$$

Our aim is to show that $\bar{c}$ as a solution to (2.4) is an equilibrium solution to (1.1). This will be proved at the end of section 4. If we denote the right-hand side of (2.4) as $A(\bar{c})$, we obtain

$$|A\bar{c}_1 - A\bar{c}_2| \le \frac{1}{aN+kM}(\frac{a}{2}|\bar{c}_1-\bar{c}_2|*|\bar{c}_1+\bar{c}_2| + \frac{kx}{2}|\bar{c}_1-\bar{c}_2|*|\bar{c}_1+\bar{c}_2|$$
$$+\frac{g}{2}|x\bar{c}_1 - x\bar{c}_2|*|x\bar{c}_1+x\bar{c}_2| + |\bar{c}_1-\bar{c}_2|*b).$$

Let us consider the operator A as a mapping of the Banach space $C[0,\alpha]$ onto itself. Then we obtain

$$\|A\bar{c}_1 - A\bar{c}_2\| \le \|\bar{c}_1-\bar{c}_2\|\frac{\alpha}{aN+kM}\left[\frac{1}{2}\left(a+\alpha k+\alpha^2 g\right)\|\bar{c}_1+\bar{c}_2\| + b\right].$$

Hence, the operator A is contractive if

$$\|\bar{c}\| < \frac{aN + kM - \alpha b}{\alpha(a + \alpha k + \alpha^2 g)} \stackrel{\text{def}}{=} R_\alpha.$$

To use the contraction mapping theorem [11] we have to check if the ball $B(R_\alpha)$ is invariant. We obtain

$$\|A\bar{c}\| \leq \frac{1}{2(aN + kM)} \left[(\alpha a + \alpha^2 k + \alpha^3 g)\|\bar{c}\|^2 + 2bN + 2\alpha b\|\bar{c}\|\right].$$

By solving the inequality $\|A\bar{c}\| \leq \|\bar{c}\|$ we may see that the ball $B(R_\alpha)$ remains invariant if

$$\|\bar{c}\| \leq \frac{aN + kM - \alpha b + \sqrt{(aN + kM - \alpha b)^2 - 2\alpha bN(a + \alpha k + \alpha^2 g)}}{\alpha(a + \alpha k + \alpha^2 g)} \tag{2.5}$$

whence we obtain that the square root expression should be non-negative. This condition with $R_\alpha > 0$ allows us to find a suitable value of α. Now we are in a position to prove the following lemma:

Lemma 2.1 *Let α satisfy the condition $R_\alpha > 0$ and be such that the square root in (2.5) is non-negative. Then there exists a unique solution to (2.4) on the interval $[0, \alpha]$ which is continuous and belongs to the ball $B(R_\alpha)$.*

Proof. Existence and uniqueness of a continuous solution $\bar{c}$ in the ball $B(R_\alpha)$ follows from the contraction mapping theorem. We now prove uniqueness for all solutions (not necessarily inside $B(R_\alpha)$). Suppose that there exists another solution $\bar{e}$ to (2.4). Its continuity follows from its integrability and we remark that the operator A maps any integrable function to a continuous one. Let us consider the restriction of $\bar{e}$ to an interval $[0, \varepsilon], \varepsilon < \alpha$. Choosing ε small enough we find that the ball $B(R_\varepsilon)$ contains two solutions $\bar{c}$ and $\bar{e}$. (Actually, R_ε tends to infinity as $\varepsilon \to 0$.) This result contradicts the uniqueness in this ball. This proves Lemma 2.1.

Our next step is to extend the solution obtained to the interval $[0, \infty)$.

Lemma 2.2 *There exists a unique continuous solution to (2.4) for all $x \geq 0$.*

Proof. Let us consider the operator A as a mapping $A : C[\alpha, 2\alpha] \to C[\alpha, 2\alpha]$ and denote by $d(x)$ a solution of (2.4) on $[\alpha, 2\alpha]$. The function $d(x)$

obeys the equality

$$d(x) = \frac{1}{aN + kM + x(b/2 + kN + gM)} \Bigg[(a+kx) \int_{\alpha}^{x} d(y)\bar{c}(x-y)dy + \frac{a+kx}{2} \int_{x-\alpha}^{\alpha} \bar{c}(x-y)\bar{c}(y)dy + g \int_{\alpha}^{x} y(x-y)d(y)\bar{c}(x-y)dy + \frac{g}{2} \int_{x-\alpha}^{\alpha} y(x-y)\bar{c}(x-y)\bar{c}(y)dy + b \left(N - \int_{0}^{\alpha} \bar{c}(x)dx - \int_{\alpha}^{x} d(y)dy \right) \Bigg]. \quad (2.6)$$

Here the function $\bar{c}$ is the solution to (2.4) on $[0, \alpha]$. Its existence and uniqueness were proved in Lemma 2.1. By standard results on integral equations, the linear Volterra equation (2.6) has a unique continuous solution $d(x)$ on the interval $[\alpha, 2\alpha]$. Put $\bar{c}(x) = d(x)$ if $\alpha < x \leq 2\alpha$. Obviously, $\bar{c}$ satisfies (2.4) for all $x \in [0, 2\alpha]$. Its continuity follows from the proof of Lemma 2.1. Now we can analogously extend the solution obtained to $[2\alpha, 4\alpha]$ and so on. From uniqueness on $[\alpha, 2\alpha]$ it follows also that the solution constructed has no branch points, otherwise we can choose b on a branch point. This completes the proof of Lemma 2.2.

Remark 2.1 *It follows from the proof of Lemma 2.1 that the function $\bar{c}$ is infinitely many times differentiable.*

Remark 2.2 *The integrability and positivity of $\bar{c}$ are not proved yet. These properties will be discussed later. It is worth pointing out the importance of the non-zero term bN in the numerator of the right-hand side of (2.4). If we had replaced bN with $b\int_0^\infty \bar{c}(x)dx$ then the contractions $A\bar{c}$ would tend to the trivial zero steady solution and we would not obtain the nontrivial solution by this approach. If $b = 0$ then by our uniqueness result only the zero continuous equilibrium solution is possible. It is also worth pointing out that the continuity condition is essential, because there are examples of nonzero discontinuous steady solutions for the pure coagulation equation [9].*

3 Strong Linear Stability

Equilibrium solutions to (2.4) are denoted by $\bar{c}(x)$. Let us assume that $g = 0$, that is we further consider kernels of the form

$$K(x, y) = a + k(x + y), \qquad F = b. \quad (3.1)$$

In this case there is no prohibition for equilibria for any $M > 0$, as is pointed out in section 2. In accordance with Theorem 1 from [10] the initial value problem (1.1),(1.2) has a mass conserving non-negative solution $c(x,t)$ if the initial function c_0 is continuous and has bounded moments. Therefore $c(x,t)$ can converge as $t \to \infty$ to the equilibrium with the same total mass M:

$$\int_0^\infty x c_0(x) dx = \int_0^\infty x \bar{c}(x) dx = M. \tag{3.2}$$

This reason forces us to consider the case $g = 0$, otherwise we cannot warrant the mass conservation law. Below we use the simplified notation $N(t)$ instead $N_0(t)$. Let us show that

$$N(t) \to N \quad \text{as} \quad t \to \infty. \tag{3.3}$$

The integration (1.1) yields

$$\frac{dN(t)}{dt} = -\frac{a}{2} N^2(t) - kMN(t) + \frac{b}{2} M. \tag{3.4}$$

By using (2.3) and solving (3.4) we obtain

$$|N(t) - z_1| \cdot |N(t) - z_2| = |N(0) - z_1| \cdot |N(0) - z_2| \exp(-\frac{1}{2} at)$$

where the constants z_1, z_2 are the roots of the quadratic equation

$$\frac{1}{2} az^2 + kMz - \frac{1}{2} bM = 0.$$

Hence, we obtain (3.3). We additionally see that if $N(0)$ satisfies (2.3) then $N(t) = N$ for all $t \geq 0$. The value of M in (2.3) is defined by the initial distribution c_0 in (3.2).
To examine the general convergence of the solution $c(x,t)$ to $\bar{c}(x)$ where $\bar{c}$ is the solution of (2.4), the function $f = c - \bar{c}$ is introduced. The substitution of $f(x,t)$ into (1.1) using (2.4),(3.1),(3.2) gives us

$$\frac{\partial f}{\partial t} = (a + kx)(\frac{1}{2} f * f + f * \bar{c} - fN(t))$$
$$-kfM - \frac{1}{2} bfx - b * f + (b - a\bar{c} - kx\bar{c})(N(t) - N) \tag{3.5}$$

with $f(x,0) = f_0(x) = c_0(x) - \bar{c}(x)$. Our main aim now is to show that $f(x,t) \to 0$ as $t \to \infty$.

If we consider $u(x,t)$ to be a linear perturbation of $f(x,t)$ then (3.5) can be linearised giving

$$u_t = (a+kx)u * \bar{c} - auN(t) - kxuN - kuM - \frac{1}{2}bxu - b * u, \qquad (3.6)$$
$$u(x,0) = u_0(x).$$

Taking the Laplace transform of (3.6) we come to the partial differential equation for the Laplace transform $U(p,t)$ of $u(x,t)$:

$$U_t + (k\bar{C} - kN - \frac{1}{2}b)U_p = (a\bar{C} - k\bar{C}_p - \frac{b}{p} - aN(t) - kM)U. \qquad (3.7)$$

By the substitution

$$U = \exp(-a\int_0^t N(s)ds - kMt)W \qquad (3.8)$$

we obtain from (3.7)

$$W_t + (k\bar{C} - kN - \frac{1}{2}b)W_p = (a\bar{C} - k\bar{C}_p - \frac{b}{p})W. \qquad (3.9)$$

The characteristic equation for (3.9) is of the form

$$dt = \frac{dp}{k\bar{C}(p) - kN - b/2} = \frac{dW}{(a\bar{C}(p) - k\bar{C}(p) - b/p)W}. \qquad (3.10)$$

By solving (3.10) and denoting for a fixed $p_0 \geq 0$

$$I(p) = \int_{p_0}^{p} \frac{dq}{k\bar{C}(q) - kN - b/2} \quad ,$$

we obtain, with (3.8) taken into account,

$$U(p,t) = \exp(-a\int_0^t N(s)ds - kMt)U_0(i(p,t)) \cdot \exp\left(\int_p^{i(p,t)} \frac{a\bar{C}(q) - k\bar{C}_p(q) - b/q}{kN + b/2 - k\bar{C}(q)}dq\right) \qquad (3.11)$$

where

$$i(p,t) = I^{-1}(I(p) + t).$$

Here I^{-1} is the inverse function of I. The existence of I^{-1} is warranted by the increasing monotonicity of the function I. For any fixed $t > 0$ the integral $\int_p^{i(p,t)}$ in (3.11) decreases in p for all $p > p_0$ due to the decreasing of $\bar{C}$ and $-\bar{C}_p$. Increasing of both functions I and I^{-1} means that the decreasing of $U_0(i(p,t))$ in p is not less than the decreasing of $U_0(p)$ because $i(p,t) \geq p$. Therefore there exists an inverse Laplace transform $u(x,t)$ of the right-hand side in (3.11) which is the solution of (3.6) and we have for a positive constant G the following estimate:

$$\|u(.,t)\|_C \leq G \exp\left(-a\int_0^t N(s)ds - kMt\right)\|u_0\|_C \tag{3.12}$$

where norms are from the space $C[0,B]$ for any fixed $0 < B < \infty$. The constant G depends on B but does not depend on t. Hence $u(x,t) \to 0$ strongly in $C[0,B]$ as $t \to \infty$, that is, the equilibrium solution $\bar{c}$ is (exponentially) strongly asymptotically stable in $C[0,B]$.

Remark 3.1 *We need to consider $B < \infty$ because we do not know at this point whether the function $\bar{c}$ belongs to the space $L^1[0,\infty)$ or $L^\infty[0,\infty)$. We prove these important properties of $\bar{c}$ in the next section.*

Example. For the simple case $k = 0$, $a = b = 1$ with $\bar{C}(p) = (p+\lambda)^{-1}$ we find

$$I(p) = 2(p - p_0), \qquad i(p,t) = p + t/2$$

and (3.11) is replaced by

$$U(p,t) = U_0(p + t/2)\exp(-t/\lambda)\frac{p^2(p+t/2+\lambda)^2}{(p+t/2)^2(p+\lambda)^2}.$$

The inverse Laplace transform gives us

$$u(x,t) = \exp(-\frac{1}{2}xt - t/\lambda)u_0(x) - \lambda t\exp(-\frac{1}{2}xt - t/\lambda)$$
$$\times\left[u_0(x) * (A(t) + xB(t))\exp(-\lambda x) + u_0(x) * (B(t) - A(t))\exp(-\frac{1}{2}xt)\right]$$

where

$$A(t) = (t/2-\lambda)^{-1} + \frac{1}{2}\lambda t(t/2-\lambda)^{-3},$$
$$B(t) = -\frac{1}{4}\lambda t(\lambda - t/2)^{-2}$$

provided $t \neq 2\lambda$. If $t = 2\lambda$ then we obtain

$$u(x,2\lambda) = \exp(-\lambda x)\Big[u_0(x) - 2\lambda^2 u_0(x) * (x\exp(-\lambda x))$$
$$+ \frac{1}{6}\lambda^4 u_0(x) * (x^3\exp(-\lambda x))\Big].$$

For this example the estimate (3.12) is more descriptive.

4 Nonlinear Estimates for Solutions

We are now ready to exlpoit the estimate (3.12). Let us denote $u(x,t) = T_t u_0(x)$ where $u(x,t)$ is the solution of equation (3.6) and T_t is the resulting semigroup operator. From the inequality (3.6) we obtain for the usual semigroup norm

$$\|T_t\| = \sup_{\|u_0\|_C \le 1} \|T_t u_0\|_C \quad \le G\exp(-a\int_0^t N(s)ds - kMt) \le G\exp(-\nu t), \tag{4.1}$$

$$0 < \nu \le kM + a\inf_{t>0} t^{-1}\int_0^t N(s)ds = kM + a\min\{N(0), N\} \tag{4.2}$$

The nonlinear initial value problem (3.5) can now be written in integral form (similar to the case in [8]) as

$$f(x,t) = T_t f_0 + \int_0^t T_{t-s}\left[\frac{1}{2}(a+kx)f*f(.,s) \right. \\ \left. +(b - a\bar{c} - kx\bar{c} - kxf(.,s))(N(s)-N)\right] ds. \tag{4.3}$$

We now introduce the norm

$$\|f\|_\nu = \sup_{t\ge 0} \exp(\nu t)\|f(.,t)\|_C. \tag{4.4}$$

If the right-hand side of the equation (4.3) is denoted by $D(f(.,t))$ then clearly for any fixed $t \ge 0$ D maps $C[0,B]$ into itself. Expressions (4.1) and (4.3) yield

$$\|D(f(.,t))\|_C \le G\exp(-\nu t)(\|f_0\|_C + \int_0^t \exp(\nu s)(\frac{1}{2}(a+kB)B\|f(.,s)\|_C^2 \\ + \sup_{x\in[0,B]} |b - a\bar{c} - kx\bar{c}| \cdot |N(s) - N| + kx\|f(.,s)\|_C |N(s)-N|)ds). \tag{4.5}$$

Multiplying (4.5) by $\exp(\nu t)$ we establish the correlation

$$\|D(f)\|_\nu \;\le\; G\|f_0\|_C + \frac{GB}{2\nu}(a+kB)\|f\|_\nu^2 + Ga_1 + a_2\|f\|_\nu. \tag{4.6}$$

where

$$a_1 = \sup_{x\in[0,B]} |b - a\bar{c} - kx\bar{c}| \int_0^\infty \exp(\nu s)|N(s) - N|ds$$

and

$$a_2 = GkB\int_0^\infty |N(s) - N|ds.$$

From (4.5) it is possible to reveal that if

$$\|f_0\|_C + a_1 \leq \frac{\nu(1-a_2)^2}{2G^2B(a+kB)} \tag{4.7}$$

and

$$a_2 < 1 \tag{4.8}$$

then the mapping D has an invariant ball in $C[0, B]$ with radius η satisfying $\eta_1 \leq \eta \leq \eta_2$ where η_1 and η_2 are the real positive roots of the quadratic equation

$$\frac{GB}{2\nu}(a+kB)z^2 - (1-a_2)z + G(\|f_0\|_C + a_1) = 0. \tag{4.9}$$

In fact, if $\|f\|_\nu \leq \eta$ for some $\eta \in [\eta_1, \eta_2]$, then from (4.6) we obtain

$$\|D(f)\|_\nu \leq G\|f_0\|_C + \frac{GB}{2\nu}(\alpha + \delta B)\eta^2 + Ga_1 + a_2\eta \leq \eta \tag{4.10}$$

which follows from the facts that $\eta_1 \leq \eta_2$ and the conditions (4.6)-(4.8) hold. We now try to find conditions for D to be a contraction in $C[0, B]$. For any f_1 and f_2 it follows from (4.1) and (4.3) that

$$\begin{aligned}
&\|D(f_1) - D(f_2)\|_C \leq \\
&\frac{1}{2}G(a+kB)\int_0^t \exp(-\nu(t-s))\|(f_1 - f_2) * (f_1 + f_2)\|_C ds \\
&+kGB\int_0^t \exp(-\nu(t-s))|N(s) - N|\|f_1 - f_2\|_C ds \\
&\leq \frac{BG}{2\nu}(a+kB)\exp(-\nu t)\|f_1 - f_2\|_\nu(\|f_1\|_\nu + \|f_2\|_\nu) \\
&+a_2 \exp(-\nu t)\|f_1 - f_2\|_\nu.
\end{aligned} \tag{4.11}$$

If the functions f_1 and f_2 belong to a ball with radius η, that is, $\|f_1\|_\nu \leq \eta$ and $\|f_2\|_\nu \leq \eta$, then from (4.10) we obtain

$$\|D(f_1) - D(f_2)\|_\nu \leq \left(\frac{BG\eta}{\nu}(a+kB) + a_2\right) \cdot \|f_1 - f_2\|_\nu. \tag{4.12}$$

Thus the mapping D is a contraction mapping in the ball with radius

$$\eta < \frac{(1-a_2)\nu}{BG(a+kB)} \stackrel{def}{=} \eta_0.$$

From equation (4.9) η_1 and η_2 are given by

$$\eta_{1,2} = \frac{(1-a_2)\nu}{BG(a+kB)} \cdot \left(1 \pm \sqrt{1 - \frac{2G^2 B(\alpha+\delta B)(\|f_0\|_C + a_1)}{\nu(1-a_2)^2}}\right)$$

and hence the bound of contraction belongs to the closed interval $[\eta_1, \eta_2]$. From standard arguments using (4.9),(4.11) and the contraction mapping theorem (e.g.[11]) we see that there exists a solution of the initial value problem (3.5) which is unique in the ball of radius $\|f\|_\nu \le \eta_0$ and belongs to the ball of radius $\|f\|_\nu \le \eta_1 < \eta_0$. Moreover, this solution tends to zero not slower than $\exp(-\nu t)$.

From the non-negativity of $c(x,t)$ as a solution to (1.1),(1.2) and its trend to $\bar{c}(x)$ we can easily see that the function $\bar{c}$ is non-negative. Using the mass conservation law and the non-negativity of the functions $\bar{c}$ and $c(x,t)$ we now see that $\bar{c}$ is integrable with weight x on all $[0,\infty)$ and that its first moment is not more tham M. By integrating (2.4) directly, we find that

$$\int_0^\infty \bar{c}(x)dx = N$$

otherwise the right-hand side of (2.4) cannot be integrated. Taking (2.3) into account we also obtain that

$$\int_0^\infty x\bar{c}(x)dx = M. \tag{4.13}$$

Therefore the function $\bar{c}$ is indeed the solution to (2.2) with the kernels (3.1). Applying Remark 2.1 we come to the following theorem.

Theorem 1 *Let the kernels K and F satisfy (3.1). Then for each $M \ge 0$ there exists a unique non-negative equilibrium solution to (1.1) such that (4.13) holds. This solution is infinitely many times differentiable.*

We can now prove the following theorem:

Theorem 2 *Let the conditions of Theorem 1 from [10] hold and kernels K and F satisfy (3.1). Suppose*

$$\int_0^\infty xc_0(x)dx = M.$$

Then any time-dependent solution (the existence of which has been proved in [10]) tends to the equilibrium solution $\bar{c}(x)$ which satisfies the equality

(4.13) provided that the estimates (4.7) and (4.8) hold. The convergence takes place in $C[0, B]$ *for any* $0 < B < \infty$ *and in* $L^1[0, \infty)$ *as* $t \to \infty$*. The rate of the convergence is proportional to* $\exp(-\nu t)$ *where* ν *is defined in (4.2).*

Remark 4.1 *The estimates (4.7) and (4.8) mean smallness of the difference between the initial function and the equilibrium.*

Proof of Theorem 2. Convergence in $C[0, B]$ was proved above. To prove convergence in $L^1[0, \infty)$ it suffices to note that "tails" of the integral of $c(x, t)$ are bounded uniformly in t due to mass conservation of time-dependent solutions [10]. Namely, we have

$$\int_B^\infty c(x,t)dx \leq \frac{1}{B}\int_0^\infty xc(x,t)dx = \frac{1}{B}\text{const.} \qquad (4.14)$$

Hence, employing (4.13), (4.14) we obtain

$$\int_0^\infty |c(x,t) - \bar{c}(x)|dx \leq \int_0^B |c(x,t) - \bar{c}(x)|dx + \frac{1}{B}\text{const.}$$

By increasing the constant B we obtain the desired result, which completes the proof of Theorem 2.

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