Existence, Uniqueness and Mass Conservation for the Coagulation–Fragmentation Equation

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Abstract

We prove global existence and uniqueness to the initial value problem for the coagulation-fragmentation equation for an unbounded coagulation kernel with possible linear growth at infinity and a fragmentation kernel from a very large class of unbounded functions. We show that the solutions satisfy the mass conservation law.

1 Introduction

We examine the general coagulation-fragmentation equation which can be written as

$$\begin{aligned}\frac{\partial}{\partial t}c(x,t) &= \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy \\ &\quad -c(x,t)\int_0^\infty K(x,y)c(y,t)dy - \frac{1}{2}c(x,t)\int_0^x F(x-y,y)dy \\ &\quad + \int_0^\infty F(x,y)c(x+y,t)dy, \qquad (1.1)\end{aligned}$$

$$c(x,0) = c_0(x) \geq 0. \qquad (1.2)$$

Equation (1.1) describes the time evolution of particles $c(x,t) \geq 0$ of mass $x \geq 0$ at time $t \geq 0$ whose change in mass is governed by the non-negative reaction rates K and F which are called, respectively, the coagulation and fragmentation kernels. These kernels are assumed to be continuous non-negative symmetric functions and are chosen to reflect the particular process being modelled. The coagulation kernel K models the rate at which particles of size x coalesce with those of size y while the kernel F expresses the rate at which particles of size $(x+y)$ fragment into those of sizes x and y.

From the physical point of view it is clear that K and F must be symmetric: $K(x,y) = K(y,x)$, $F(x,y) = F(y,x)$ for all $0 \leq x, y < \infty$. The first and fourth integrals in (1.1) describe the growth of the number of particles of size x due to coagulation and fragmentation respectively, while the second and third integrals describe the reverse of these processes. From physical considerations all the functions in (1.1) have to be non-negative. A brief physical interpretation of the integrals appearing on the right hand side of equation (1.1) can be found in Drake [4], Melzak [16] or Stewart [19]. Applications of (1.1) can be found in many problems including chemistry (e.g. reacting polymers), physics (aggregation of colloidal particles, growth of gas bubbles in solids), astrophysics (formation of stars and planets) and meteorology (merging of drops in atmospheric clouds).

Equation (1.1) is similar to the Boltzmann equation of gas kinetics but unlike the Boltzmann equation the problem concerned here obeys in general only the mass conservation law which is expressed by the first moment

$$\int_0^\infty xc(x,t)dx = \text{const.}$$

The general coagulation–fragmentation equation has no estimates such as the energy conservation law which corresponds to the second moment

$$\int_0^\infty x^2 c(x,t)dx = \text{const.}$$

In addition, if the rate of growth of the coagulation kernel is high enough, infringement of the mass conservation law occurs. It has been proved by McLeod [14],[15] that for the unbounded kernel $K = xy$, with $F = 0$, the mass conservation law for solutions breaks down at a finite time when the second moment has blow-up. Leyvraz and Tschudi [13], Ernst, Ziff and Hendriks [8] and Galkin [10] showed that for this kernel there is a global time solution. Recently, McLeod's approach was reconsidered by Slemrod

[17] with another derivation of a solution existing after the infringement of the mass conservation law. It is known that if the solution satisfies

$$\int_0^\infty x^2 c(x,t)dx < \infty,$$

then for constant coagulation and fragmentation kernels the solutions to (1.1) are mass conserving [1]. Conditions ensuring mass conservation have also been derived in [2, 22].

For the discrete case of equation (1.1) existence of the mass conserving solution was proved by Ball and Carr [2] for coagulation kernels with at most linear growth at infinity. For a subclass of coagulation and fragmentation kernels they also succeeded to demonstrate uniqueness. We should observe that an essential mathematical difference between discrete and continuous models of coagulation-fragmentation consists in the fact that the space l^1 is contained in l^∞ for the discrete case which is not true in the continuous case. Therefore the continuous version (which we treat in this paper) ought to include additional estimates.

For both nonzero kernels K and F the global existence and uniqueness of solutions to (1.1) has been investigated in [1, 2, 3, 5, 12, 16, 18, 19, 20]. For the case $K = a, F = b$ with a, b constants the convergence to equilibrium of solutions has been studied via a Lyapunov function by Aizenman and Bak [1] and by the authors [23]. It is our aim to prove existence, uniqueness and mass conservation theorems for the initial value problem (1.1), (1.2) with $K \leq k(1+x+y)$ and a very large class of fragmentation kernels including bounded ones; in this paper we use and generalize the approach of [9, 11]. In [6] we prove the existence of an equilibrium solution for linear coagulation and constant fragmentation kernels and demonstrate convergence to equilibrium for any time-dependent solutions possessing initial data near the equilibrium distribution.

2 Existence theorem

We introduce some functional spaces. Firstly, we fix a positive constant $T > 0$. Let Π be the strip

$$\Pi = \{(x,t) : \ x \in [0,\infty), 0 \leq t \leq T\}$$

and $\Pi(X)$ be the rectangle

$$\Pi(X) = \{(x,t) : \ 0 \leq x \leq X, 0 \leq t \leq T\}.$$

We denote by $\Omega_\lambda(T)$ and $\Omega_{0,r}(T)$ the spaces of continuous functions f with bounded norms

$$\|f\|_\lambda = \sup_{0\le t\le T} \int_0^\infty \exp(\lambda x)|f(x,t)|dx$$

and

$$\|f\|_{0,r} = \sup_{0\le t\le T} \int_0^\infty x^r|f(x,t)|dx, \quad r \ge 1.$$

Let

$$\Omega(T) = \bigcup_{\lambda>0} \Omega_\lambda(T).$$

It should be noted that the following inclusions take place

$$\Omega_{\lambda_1} \supset \Omega_{\lambda_2},\ \lambda_1 < \lambda_2; \qquad \Omega_0 \stackrel{\text{def}}{=} \Omega_{0,0} \supset \Omega_{0,1} \supset \cdots \supset \Omega_{0,r} \supset \cdots \supset \Omega.$$

$\Omega(T)$ may be equipped with the topology of the inductive limit of topologies in $\Omega_\lambda(T)$. Cones of non-negative functions in $\Omega_{0,r}(T), \Omega_\lambda(T)$ and $\Omega(T)$ are denoted $\Omega^+_{0,r}(T)$, $\Omega^+_\lambda(T)$ and $\Omega(T)^+$ respectively. In this section we prove the following theorem.

Theorem 1 *Let the functions $K(x,y)$ and $F(x,y)$ be continuous, non-negative and symmetric. Suppose also that*

$$K(x,y) \le k(1+x+y), \quad k>0 \tag{2.1}$$

and there exist positive constants m, m_1 and b such that

$$\int_0^x F(x-y,y)dy \le b(1+x^{m_1}), \quad F(x-y',y) \le b(1+x^m), \tag{2.2}$$

$$0 \le y \le y' \le x, \quad x \in [0,\infty).$$

Let the initial data function satisfy either:
1^0 $\quad c_0 \in \Omega^+_{0,r}(0),\ r > max\{m,1\}, \quad$ *and* $r \ge m_1$*;*
or
2^0 $\quad c_0 \in \Omega^+(0)$.
Then the problem (1.1),(1.2) has, respectively, either:
1^0 *at least one solution in* $\Omega^+_{0,r}(T)$*;*
or
2^0 *a solution in* $\Omega^+(T)$*.*

We begin the proof of Theorem 1 with some preliminary results.

Lemma 2.1 *Let the conditions of Theorem 1 hold and, in addition, suppose the functions $K(x,y)$ and $F(x,y)$ have compact support. Then there exists a solution $c(x,t)$ to the initial value problem (1.1),(1.2) such that*
1^0 $\quad c \in \Omega^+_{0,r}(T)$
or
2^0 $\quad c \in \Omega^+(T)$
respectively. This solution obeys the mass conservation law and is unique in the class of continuous functions having bounded first moment.

Proof. Existence of a continuous non-negative solution follows from [19], its uniqueness follows from [20]. Since the kernels K and F have a compact support, the "tail" of the solution does not change in time and coincides with the "tail" of c_0. Hence, $c(x,t)$ belongs to $\Omega^+_{0,r}(T)$ or $\Omega^+(T)$ respectively. This proves Lemma 2.1.

When K and F belong to class (2.1), (2.2) we construct a sequence of continuous kernels $\{K_n, F_n\}_{n=1}^{\infty}$ from the class (2.1), (2.2) with compact support for each $n \geq 1$, such that

$$K_n(x,y) = K(x,y), 0 \leq x, y \leq n, n \geq 1, \tag{2.3}$$

$$F_n(x,y) = F(x,y), 0 \leq x, y \leq n, n \geq 1, \tag{2.4}$$

$$K_n(x,y) \leq K(x,y), 0 \leq x, y < \infty, n \geq 1, \tag{2.5}$$

$$F_n(x,y) \leq F(x,y), 0 \leq x, y < \infty, n \geq 1. \tag{2.6}$$

In accordance with Lemma 2.1, the sequence $\{K_n, F_n\}_{n=1}^{\infty}$ generates on Π a sequence $\{c_n\}_{n=1}^{\infty}$ of non-negative continuous solutions to the problem (1.1), (1.2) with the kernels K_n, F_n. These solutions belong to $\Omega^+_{0,r}(T)$ or $\Omega^+(T)$ respectively.
Let us denote the r-th moment of the functions c_n as

$$N_{r,n}(t) = \int_0^{\infty} x^r c_n(x,t)dx, r \geq 0, n \geq 1.$$

By direct integration of (1.1) with the weight x , we obtain the mass conservation law

$$N_{1,n}(t) = N_1 = \text{const}, n \geq 1, t > 0. \tag{2.7}$$

All the integrals exist due to the compact support of the kernels. Integrating (1.1) with the weight x^2 and using (2.1), we also obtain

$$\frac{dN_{2,n}(t)}{dt} \leq kN_1^2 + 2kN_1N_{2,n}(t).$$

Hence, $N_{2,n}(t)$ is bounded on $t \in [0,T]$:

$$N_{2,n}(t) \le \bar{N}_2, 0 \le t \le T,\ n \ge 1. \tag{2.8}$$

Similarly, step by step, we obtain the uniform boundedness of $N_{r,n}(t)$ with respect to $n \ge 1,\ 0 \le t \le T$. The uniform boundedness of the zero moment $N_{0,n}$ follows via (2.2) from the inequalities

$$\frac{dN_{0,n}}{dt} \le \frac{1}{2}\int_0^\infty c_n(x,t)\int_0^x F(x-y,y)dydx \le \frac{1}{2}b(N_{0,n}+N_{m_1,n})$$

and the condition $m_1 \le r$. Consequently,

$$N_{k,n}(t) \le \bar{N}_k = \text{const if } t \in [0,T],\ n \ge 1,\ 0 \le k \le r. \tag{2.9}$$

We are now in a position to formulate the following Lemma:

Lemma 2.2 *The sequence* $\{c_n\}_{n=1}^\infty$ *is relatively compact in the uniform-convergence topology of continuous functions on each rectangle* $\Pi(X)$.

Proof. **Step 1.** We first prove that $\{c_n\}_{n=1}^\infty$ is uniformly bounded on $\Pi(X)$. Since solutions c_n of (1.1),(1.2) with kernels K_n, F_n are non-negative then by virtue of (2.1),(2.4),(2.6),(2.9) we obtain for $(x,t) \in \Pi(X)$:

$$c_n(x,t) \le \bar{c}_0 + \int_0^t \left(\frac{1}{2}k(1+X)c_n * c_n(x,s) + b(\bar{N}_0+\bar{N}_r)\right)ds. \tag{2.10}$$

Here $\bar{c}_0 = \sup_{0\le x\le X} c_0(x)$ and $f*g$ is the convolution,

$$f*g(x) = \int_0^x f(x-y)g(y)dy.$$

We define the "upper" function for the integral inequality (2.10) to be

$$g(x,t) = g_0 + \int_0^t \left(\frac{1}{2}k(1+X)g*g(x,s) + g(x,s)\right)ds, \tag{2.11}$$
$$0 \le t \le T, 0 \le x < \infty,$$

where $g_0 = \max\{\bar{c}_0, b(\bar{N}_0+\bar{N}_r)\} = \text{const}$. Taking the Laplace transform of this relation with respect to x, we obtain

$$g(x,t) = g_0 \exp\left(\frac{1}{2}g_0 kx(1+X)(\mathrm{e}^t-1)+t\right), 0 \le t \le T, 0 \le x < \infty. \tag{2.12}$$

Our next aim is to prove that the inequality

$$c_n(x,t) \le g(x,t) \text{ for } (x,t) \in \Pi(X)$$

holds for each integer $n \ge 1$.
We introduce the auxiliary function

$$g_\varepsilon(x,t) = g_0 + \varepsilon + \int_0^t \left(\frac{1}{2}k(1+M) g_\varepsilon * g_\varepsilon(x,s) + g_\varepsilon(x,s)\right) ds, \qquad (2.13)$$

$$(x,t) \in \Pi, \varepsilon > 0.$$

Clearly $c_n(x,0) < g_\varepsilon(x,0)$ for $0 \le x \le X$. We assume that, for some $n \ge 1$, there is a set D of points $(x,t) \in \Pi(X)$ on which $c_n(x,t) = g_\varepsilon(x,t)$. Since D does not contain points on the coordinate axes, we choose $(x_0,t_0) \in D$ so that the rectangle $Q = [0,x_0) \times [0,t_0)$ contains no points of D. Since g_ε and c_n are continuous, we have $c_n(x,t) < g_\varepsilon(x,t)$ for $(x,t) \in Q$. The values of c_n and g_ε coincide at the point (x_0,t_0). Hence

$$c_n(x_0,t_0) = g_\varepsilon(x_0,t_0) > g_0+\varepsilon+\int_0^{t_0} \left(\frac{1}{2}k(1+X) c_n * c_n(x_0,s) + c_n(x_0,s)\right) ds. \qquad (2.14)$$

This is proved by using the fact that the values of the arguments of g_ε in the integrand (2.13) are in Q. Combining (2.10) and (2.14) we arrive at the contradiction $c_n(x_0,t_0) > c_n(x_0,t_0)$, which proves that D is empty and

$$c_n(x,t) < g_\varepsilon(x,t), \qquad (x,t) \in \Pi(X),\ n \ge 1.$$

Using (2.12) we have the continuity of g_ε as a function of ε. Letting ε tend to zero we find that actually

$$c_n(x,t) \le g(x,t) \quad \text{for} \quad (x,t) \in \Pi(X), n \ge 1,$$

and hence the sequence $\{c_n\}_{n=1}^{\infty}$ is bounded uniformly on $\Pi(X)$:

$$0 \le c_n(x,t) \le g_0 \exp\left(\frac{1}{2} g_0 k X (1+X)(e^T - 1) + T\right) = M_1 = \text{const.} \qquad (2.15)$$

Step 2. We show the equicontinuity of $\{c_n\}_{n=1}^{\infty}$ with respect to t. From (1.1) we note that for $0 \le t \le t' \le T, 0 \le x \le X, n \ge 1$ the following inequality takes place

$$|c_n(x,t') - c_n(x,t)| \le \int_t^{t'} \left\{\frac{1}{2}\int_0^x K_n(x-y,y) c_n(x-y,s) c_n(y,s) dy \right.$$
$$+ c_n(x,s)\int_0^\infty K_n(x,y) c_n(y,s) dy + \int_0^\infty F_n(x,y) c_n(x+y,s) dy$$
$$\left. + \frac{1}{2} c_n(x,s) \int_0^x F_n(x-y,y) dy \right\} ds. \qquad (2.16)$$

It follows from (2.5),(2.6) and (2.15) that the first and the fourth terms of the integrand in (2.16) are uniformly bounded. The second and the third terms in (2.16) are uniformly bounded by virtue of the uniform boundedness of the sequence $\{c_n\}_{n=1}^{\infty}$ on $\Pi(X)$, equations (2.3)–(2.6) and the inequalities

$$\int_0^{\infty} K_n(x,y)c_n(y,s)dy \leq k(1+X)\bar{N}_0 + kN_1, \tag{2.17}$$

$$\int_0^{\infty} F_n(x,y)c_n(x+y,s)dy = \int_x^{\infty} c_n(y,s)F(y-x,x)dy \leq b(\bar{N}_0+\bar{N}_r) \tag{2.18}$$

with $0 \leq s \leq T,\ n \geq 1$. Applying (2.17),(2.18) to (2.16), we finally obtain

$$\sup_{0\leq x\leq X} |c_n(x,t') - c_n(x,t)| \leq M_2|t'-t|, 0 \leq t \leq t' \leq T, n \geq 1. \tag{2.19}$$

The constant M_2 is independent of n and hence $\{c_n\}_{n=1}^{\infty}$ is equicontinuous with respect to the variable t on $\Pi(X)$.

Step 3. We next establish that $\{c_n\}_{n=1}^{\infty}$ is equicontinuous with respect to x. Let $0 \leq x \leq x' \leq X$; then for each $n \geq 1$ we have

$$\begin{aligned}
&|c_n(x',t) - c_n(x,t)| \leq |c_0(x') - c_0(x)| \\
&\quad + \int_0^t \Bigg\{ \frac{1}{2}\int_x^{x'} K_n(x'-y,y)c_n(x'-y,s)c(y,s)dy \\
&\quad + \frac{1}{2}\int_0^x |K_n(x'-y,y) - K_n(x-y,y)|c_n(x'-y,s)c_n(y,s)dy \\
&\quad + \frac{1}{2}\int_0^x K_n(x-y,y)\cdot|c_n(x'-y,s) - c_n(x-y,s)|c_n(y,s)dy \\
&\quad + |c_n(x',s) - c_n(x,s)| \int_0^{\infty} K_n(x',y)c_n(y,s)dy && (2.20)\\
&\quad + c_n(x,s)\int_0^{\infty} |K_n(x',y) - K_n(x,y)|c_n(y,s)dy && (2.21)\\
&\quad + \int_{x'}^{\infty} c_n(y,s)|F_n(x,y-x') - F_n(x,y-x)|dy && (2.22)\\
&\quad + \int_{x'}^{\infty} c_n(y,s)|F_n(x',y-x') - F_n(x,y-x')|dy && (2.23)\\
&\quad + \frac{1}{2}|c_n(x',s) - c_n(x,s)| \int_0^{x'} F_n(x'-y,y)dy \\
&\quad + \frac{1}{2}c_n(x,s)\int_x^{x'} F_n(x'-y,y)dy + \int_x^{x'} c_n(y,s)F_n(x,y-x)dy \\
&\quad + \frac{1}{2}c_n(x,s)\int_0^x |F_n(x'-y,y) - F_n(x-y,y)|dy \Bigg\} ds. && (2.24)
\end{aligned}$$

It follows from (2.3),(2.4) that the kernel sequence $\{K_n, F_n\}_{n=1}^{\infty}$ we have constructed is equicontinuous on each rectangle $[0, X] \times [0, z], z > 0$.

Let us remark that if $\phi(x)$ is non-negative and measurable and $\psi(x)$ is positive and nondecreasing for $x > 0$, then

$$\int_z^{\infty} \phi(x)dx \leq \frac{1}{\psi(z)} \int_0^{\infty} \phi(x)\psi(x)dx, \quad z > 0, \tag{2.25}$$

if the integrals exist and are finite.

Our aim now is to show that if the difference $|x' - x|$ is small enough, then the left-hand side of (2.24) is small also. Fix an arbitrary $\varepsilon > 0$ and choose $\delta(\varepsilon), 0 < \delta(\varepsilon) < \varepsilon$, such that

$$\sup_{|x'-x|<\delta} |c_0(x') - c_0(x)| < \varepsilon, \tag{2.26}$$

$$\sup_{|x'-x|<\delta} (|K_n(x', y) - K_n(x, y)| + |F_n(x', y) - F_n(x, y)|) < \varepsilon, \tag{2.27}$$

$$\sup_{|x'-x|<\delta} |F_n(x, y - x') - F_n(x, y - x)| < \varepsilon. \tag{2.28}$$

The inequalities (2.27) and (2.28) hold uniformly with respect to $n \geq 1$ and $0 \leq y \leq z$. The rule for choosing the constant $z = z(\varepsilon)$ is given below in expressions (2.32), (2.34). Introduce the modulus of continuity

$$\omega_n(t) = \sup_{|x'-x|<\delta} |c_n(x', t) - c_n(x, t)|,\ 0 \leq x, x' \leq X.$$

Using (2.1),(2.5),(2.6),(2.15), we can easily demonstrate the smallness of terms in (2.24) whose integrals are over finite intervals. To show the smallness of the term at (2.20) we have to use the uniform boundedness of the integral which follows from (2.1),(2.7),(2.9):

$$|c_n(x', s) - c_n(x, s)| \int_0^{\infty} K_n(x', y)c_n(y, s)dy \leq$$
$$\leq k\omega_n(s)((1 + X)\bar{N}_0 + N_1), \quad n \geq 1,\ 0 \leq x, x' \leq X.$$

The summands in the terms (2.21)-(2.23) are more complicated. Let us consider (2.21). Using the partitioning $\int_0^{\infty} = \int_0^z + \int_z^{\infty}$, we find that by

(2.1), (2.9) and (2.27)

$$\int_0^\infty |K_n(x',y) - K_n(x,y)|c_n(y,s)dy \le$$
$$\le \varepsilon \bar{N}_0 + 2k(1+X)\int_z^\infty c_n(y,s)dy + 2k\int_z^\infty yc_n(y,s)dy. \qquad (2.29)$$

Let us use (2.25) with $\phi(x) = c_n(x),\ \psi(x) = x$ or $\phi(x) = xc_n(x),\ \psi(x) = x^{r-1}$ in the second and third terms of (2.29) respectively. Also, recall equation (2.8). Then we arrive at the expressions

$$\int_z^\infty c_n(y,s)dy \le \frac{1}{z}N_1, \qquad (2.30)$$
$$\int_z^\infty yc_n(y,s)dy \le \frac{1}{z^{r-1}}\bar{N}_r. \qquad (2.31)$$

If we choose z such that

$$\frac{1}{z}N_1 \le \varepsilon \quad \text{and} \quad \frac{1}{z^{r-1}}\bar{N}_r \le \varepsilon \qquad (2.32)$$

then from (2.29)

$$\int_0^\infty |K_n(x',y) - K_n(x,y)|c_n(y,s)dy \le \text{const}\cdot\varepsilon. \qquad (2.33)$$

The same reasoning should be used to estimate the terms (2.22) and (2.23). For (2.22) we obtain

$$\int_{x'}^\infty c_n(y,s)|F_n(y-x',x) - F_n(y-x,x)|dy$$
$$\le \varepsilon\bar{N}_0 + \int_z^\infty c_n(y,s)F_n(y-x',x)dy + \int_z^\infty c_n(y,s)F_n(y-x,x)dy$$
$$\le \varepsilon\bar{N}_0 + 2b\int_z^\infty c_n(y,s)(1+y^m)dy \le \varepsilon\bar{N}_0 + 2b\frac{\bar{N}_1}{z} + 2b\frac{\bar{N}_r}{z^{r-m}}.$$

If (2.32) holds and

$$\frac{\bar{N}_0}{z^{r-m}} < \varepsilon \qquad (2.34)$$

then

$$\int_{x'}^\infty c_n(y,s)|F_n(x,y-x') - F_n(x,y-x)|dy \le \text{const}\cdot\varepsilon. \qquad (2.35)$$

Finally, using (2.15),(2.26),(2.27),(2.28),(2.33) and (2.35) we obtain from the whole inequality (2.24):

$$\omega_n(t) \le M_3 \cdot \varepsilon + M_4 \int_0^t \omega_n(s)ds, \quad 0 \le t \le T.$$

Here the positive constants M_3 and M_4 are independent of n and ε and therefore by Gronwall's inequality

$$\omega_n(t) \le M_3 \varepsilon \exp(M_4 T) \stackrel{\text{def}}{=} M_5 \cdot \varepsilon. \tag{2.36}$$

We conclude from (2.19) and (2.36) that

$$\sup_{|x'-x|<\delta, |t'-t|<\delta} |c_n(x',t') - c_n(x,t)| \le (M_2 + M_5)\varepsilon, \tag{2.37}$$

$$0 \le x, x' \le X, 0 \le t, t' \le T.$$

The assertion of Lemma 2.2 is then a consequence of (2.15),(2.37) and Arzela's theorem [7]. Lemma 2.2 has now been proved.

Proof of Theorem 1: Case 1^0.
By means of the diagonal method we select a subsequence $\{c_i\}_{i=1}^{\infty}$ from $\{c_n\}_{n=1}^{\infty}$ converging uniformly on each compact set in Π to a continuous non-negative function c. Let us consider an integral $\int_0^z x^k c(x,t)dx, \quad 0 \le k \le r$. Since for all $\varepsilon > 0$ there exists $i \ge 1$ such that

$$\int_0^z x^k c(x,t)dx \le \int_0^z x^k c_i(x,t)dx + \varepsilon \le \bar{N}_k + \varepsilon, \tag{2.38}$$

then

$$\int_0^{\infty} x^k c(x,t)dx \le \bar{N}_k, \quad 0 \le k \le r \tag{2.39}$$

because in (2.38) both z and ε are arbitrary. Similarly we obtain

$$\int_0^{\infty} xc(x,t)dx \le N_1. \tag{2.40}$$

The inequality (2.40) can be transformed into an equality giving the mass conservation law: this will be proved a below. We should show now that the function $c(x,t)$ is a solution to the initial value problem (1.1),(1.2). To prove this assertion we write the equations (1.1), (1.2) in the integral form for c_n

with K_n, F_n and change c_n, K_n, F_n to $c_n - c + c$, $K_n - K + K$, $F_n - F + F$ respectively. Then we obtain

$$
\begin{aligned}
&(c_i - c)(x,t) + c(x,t) = c_0(x) \\
&+ \int_0^t \Big\{ \frac{1}{2} \int_0^x (K_i - K)(x-y,y) c_i(x-y,s) c_i(y,s) dy \\
&+ \frac{1}{2} \int_0^x K(x-y,y)(c_i(x-y,s) - c(x-y,s)) c_i(y,s) dy \\
&+ \frac{1}{2} \int_0^x K(x-y,y)(c_i(y,s) - c(y,s)) c(x-y,s) dy \\
&+ \frac{1}{2} \int_0^x K(x-y,y) c(y,s) c(x-y,s) dy \\
&- c_i(x,s) \int_0^\infty (K_i - K)(x,y) c_i(y,s) dy \\
&- (c_i - c)(x,s) \int_0^\infty K(x,y) c_i(y,s) dy \\
&- c(x,s) \int_0^\infty K(x,y)(c_i - c)(y,s) dy - c(x,s) \int_0^\infty K(x,y) c(y,s) dy \\
&+ \int_0^\infty (F_i - F)(x,y) c_i(x+y,s) dy + \int_0^\infty F(x,y)(c_i - c)(x+y,s) dy \\
&+ \int_0^\infty F(x,y) c(x+y,s) dy - \frac{1}{2} c_i(x,s) \int_0^x (F_i - F)(x-y,y) dy \\
&- \frac{1}{2}(c_i - c)(x,s) \int_0^x F(x-y,y) dy \\
&- \frac{1}{2} c(x,s) \int_0^x F(x-y,y) dy \Big\} ds. \qquad (2.41)
\end{aligned}
$$

Passing to the limit as $i \to \infty$ in (2.41) we can see that the terms with integrals over $[0, \infty)$ tend to zero due to the estimates for their "tails", which may be obtained via (2.25) (2.39), (2.40) taking into account similar arguments in (2.33),(2.35):

$$| \int_z^\infty (K_i - K)(x,y) c_i(y,s) dy | \le \frac{2k}{z}(1+x) N_1 + \frac{2k}{z^{r-1}} \bar{N}_r), \qquad (2.42)$$

$$| \int_z^\infty K(x,y)(c_i - c)(y,s) dy | \le \frac{2k}{z}(1+x) N_1 + \frac{2k}{z^{r-1}} \bar{N}_r), \qquad (2.43)$$

$$| \int_z^\infty (F_i - F)(x,y) c_i(x+y,s) dy | \le \frac{2b}{z} N_1 + \frac{2b}{z^{r-m}} \bar{N}_r, \qquad (2.44)$$

$$| \int_z^\infty F(x,y)(c_i - c)(x+y,s) dy | \le \frac{2b}{z} N_1 + \frac{2b}{z^{r-m}} \bar{N}_r. \qquad (2.45)$$

Other difference terms in (2.41) can be easily shown to tend to zero. Finally, we find that the function c is a solution of the problem (1.1), (1.2) written in integral form:

$$c(x,t) = c_0(x) + \int_0^t \left\{ \frac{1}{2} \int_0^x K(x-y,y)c(x-y,s)c(y,s)dy - \right.$$
$$-c(x,s)\int_0^\infty K(x,y)c(y,s)dy +$$
$$\left. + \int_0^\infty F(x,y)c(x+y,s)dy - \frac{1}{2}c(x,s)\int_0^x F(x-y,y)dy \right\} ds. \qquad (2.46)$$

It follows from (2.42)–(2.45) and the continuity of $c(x,t)$ that the right-hand side in (1.1), evaluated at c, is a continuous function on Π. Differentiation of (2.46) with respect to t establishes that c is a continuous differentiable solution of (1.1),(1.2). In accordance with (2.39) it belongs to $\Omega_{0,r}$. This proves case 1^0 of Theorem 1.

Proof of Theorem 1: Case 2^0.

To prove the second case of Theorem 1 it suffices to prove that, similarly to (2.8), the functions c_n belong to $\Omega^+(T)$ uniformly, that is, there exists $\lambda > 0$ such that for all $n \geq 1,\ t \in [0,T]$

$$\int_0^\infty \exp(\lambda x) c_n(x,t)dx \ \leq \ \text{const}. \qquad (2.47)$$

Actually, in this case the uniform convergence on each compact set to $c(x,t)$ implies that $c \in \Omega^+_\lambda(T)$ with the same λ. Denote

$$\sigma_n(\lambda,t) = \int_0^\infty (\exp(\lambda x) - 1)c_n(x,t)dx.$$

Multiplying (1.1) by $\exp(\lambda x) - 1$ and taking into account the positivity of $c_n(x,t)$, we obtain

$$\frac{\partial \sigma_n}{\partial t} \leq k(\frac{1}{2}\sigma_n^2 + \sigma_n \frac{\partial}{\partial \lambda}\sigma_n - \sigma_n N_1), \qquad (2.48)$$

$$\sigma_n(\lambda,0) = \int_0^\infty (\exp(\lambda x) - 1)c_0(x)dx. \qquad (2.49)$$

Let us consider the "upper" function $\sigma(\lambda,t)$ which satisfies the following equation:

$$\frac{\partial \sigma}{\partial t} = k(\frac{1}{2}\sigma^2 + \sigma \frac{\partial \sigma}{\partial \lambda} - \sigma N_1), \qquad (2.50)$$

$$\sigma(\lambda,0) \stackrel{\text{def}}{=} \sigma_0(\lambda) > \sigma_n(\lambda,0), \lambda > 0;\ \sigma_0(0) = \sigma_n(0,0) = 0. \tag{2.51}$$

If the problem (2.50),(2.51) has a smooth enough solution then

$$\sigma_n(\lambda,t) \le \sigma(\lambda,t) \tag{2.52}$$

for $0 \le \lambda < \bar\lambda,\ \bar\lambda > 0,\ \ 0 \le t \le T$ for some $\bar\lambda$. To show this fact we use the substitution

$$\sigma_n = \exp(-kN_1 t)\alpha_n, \quad \sigma = \exp(-kN_1 t)\alpha.$$

Then from (2.48) and (2.50) we have

$$\frac{d}{dt}\alpha_n \le \frac{1}{2}k\exp(-kN_1 t)\alpha_n^2, \tag{2.53}$$

$$\frac{d}{dt}\alpha = \frac{1}{2}k\exp(-kN_1 t)\alpha^2, \tag{2.54}$$

where $\frac{d}{dt}$ in (2.53),(2.54) means differentiation along characteristics of (2.48), (2.50) respectively. Let $(\hat\lambda,\hat t)$ be the first point where $\sigma_n(\lambda,t) = \sigma(\lambda,t)$, i.e. $\sigma_n(\lambda,t) < \sigma(\lambda,t)$ for $0 \le \lambda < \bar\lambda,\ 0 \le t < \hat t$. Due to (2.51) we have $\hat t > 0$. Also, the functions σ_n increase in λ. Then we obtain the following contradiction:

$$\alpha(\hat\lambda,\hat t) = \alpha(\lambda(t),t) + \frac{1}{2}k\int_t^{\hat t}\exp(-kN_1 s)\alpha^2(\lambda(s),s)ds$$

$$> \alpha_n(\lambda_n(t),t) + \frac{1}{2}k\int_t^{\hat t}\exp(-kN_1 s)\alpha_n^2(\lambda_n(s),s)ds = \alpha_n(\hat\lambda,\hat t). \tag{2.55}$$

The first and second integrations in (2.55) are along characteristics of the equations (2.50) and (2.48) respectively. We have used the fact that $\lambda(s) > \lambda_n(s)$. The inequality (2.52) is now proved.

Our next aim is to show that there exists a solution to (2.50),(2.51), which is bounded in a neighbourhood of zero for all $0 \le t \le T$. Firstly, we formulate for convenience the following well-known lemma which is fundamental to the characteristics method.

Lemma 2.3 *Let the functions $a(z,t,u)$ and $f(t,u)$ be continuous in $R^n \times R^1_+ \times R^1$ and $R^1_+ \times R^1$ respectively and let $u(z,t)$ be a solution to the problem*

$$u_t(z,t) + a(z,t,u)u_z(z,t) = f(t,u) \tag{2.56}$$

$$u(z,0) = u_0(z), \qquad z \in R^n, \quad t \in R^1_+.$$

Let the function v be a solution to the simplified problem

$$v_t(t, v_0) = f(t, v) \qquad (2.57)$$
$$v(0, v_0) = v_0 = const.$$

Let $z_0(z,t)$ be the initial point on a characteristic for the problem (2.56) which pass through the point (z,t). Then

$$u(z,t) = v(t, u_0(z_0(z,t))). \qquad (2.58)$$

To study (2.50),(2.51) we consider the following problem:

$$\frac{\partial\sigma}{\partial t} = k(\frac{1}{2}\sigma^2 + \sigma\frac{\partial\sigma}{\partial\lambda} - g(\lambda)\sigma) + \varepsilon, \qquad (2.59)$$

$$\sigma(\lambda, 0) = \sigma_0(\lambda), \quad \lambda \geq 0, t \geq 0. \qquad (2.60)$$

Lemma 2.4 *Let $\sigma_0(\lambda) > 0$ if $\lambda > 0$, $\sigma_0(0) = 0$; $g(\lambda) = G - \delta(\lambda)$, $G = const > 0$; $\delta(\lambda) \to 0$ as $\lambda \to 0$ and $\sigma_0'(0) \leq G$. Let $\sigma_0(\lambda)$ be a holomorphic function in a neighborhood of $\lambda = 0$. Let us fix $T > 0$. Then there exist $\hat{\lambda}(T) > 0$ and $\hat{\varepsilon}(T) > 0$ such that the initial value problem (2.59),(2.60) has for $t \in [0,T]$ a unique solution for $0 \leq \lambda < \hat{\lambda}, 0 \leq \varepsilon < \hat{\varepsilon}$.*

Proof. Firstly, let $\delta(\lambda) = 0$. We consider the auxiliary problem

$$v_t = \frac{1}{2}kv^2 - kGv + \varepsilon, \quad v|_{t=0} = v_0$$

with the solution

$$v(t, v_0) = v_2 + (v_1 - v_2)\left[1 + \left(\frac{v_1 - v_2}{v_0 - v_2} - 1\right)\exp(\frac{1}{2}kt(v_1 - v_2))\right]^{-1}.$$

Here v_1 and v_2 are roots of the trinomial $\frac{1}{2}kv^2 - kGv + \varepsilon$. Choosing ε small enough, we have $v_1 >> v_2 \geq 0$. Using Lemma 2.3, we have

$$\sigma(\lambda, t) = v_2 + (v_1 - v_2)\left[1 + \left(\frac{v_1 - v_2}{\sigma_0(\lambda_0) - v_2} - 1\right)\exp(\frac{1}{2}kt(v_1 - v_2))\right]^{-1} \qquad (2.61)$$

We investigate the quantity λ_0. Let $\lambda(t)$ be a solution of the characteristic equation of the problem (2.59), (2.60):

$$d\lambda/dt = -k\sigma(\lambda, t).$$

Using (2.61), we obtain

$$\lambda(t) = \lambda_0 - k\int_0^t \{v_2 + (v_1 - v_2)$$
$$\cdot \left[1 + \left(\frac{v_1 - v_2}{\sigma_0(\lambda_0) - v_2} - 1\right)\exp(\frac{1}{2}ks(v_1 - v_2))\right]^{-1}\}ds,$$

whence

$$\lambda = \lambda_0 - kv_1 t + 2\log\left(1 + \left(\frac{v_1 - v_2}{\sigma_0(\lambda_0) - v_2} - 1\right)\exp(\frac{1}{2}kt(v_1 - v_2))\right)$$
$$-2\log\left(\frac{v_1 - v_2}{\sigma_0(\lambda_0) - v_2}\right).$$

By substituting (2.61) into the last expression, we obtain the equality:

$$\lambda_0 = \lambda + kv_1 t + 2\log\left(\frac{\sigma - v_2}{v_1 - v_2} + \left(1 - \frac{\sigma - v_2}{v_1 - v_2}\right)\exp(-\frac{1}{2}kt(v_1 - v_2))\right). \quad (2.62)$$

Using (2.61), we introduce for consideration the function

$$S(\sigma, \lambda, t) = \sigma - v_2 - (v_1 - v_2)(\sigma_0(\lambda_0) - v_2)$$
$$\cdot \left[\sigma_0(\lambda_0) - v_2 + (v_1 - \sigma_0(\lambda_0))\exp(\frac{1}{2}kt(v_1 - v_2))\right]^{-1}.$$

From (2.62) we can see that for small $\sigma,\ v_2$ and λ, the value λ_0 is small for all $t,\ 0 \le t \le T$. Consequently, the function S is analytic in the polycircle

$$\left\{(\lambda, \sigma, t):\ |\lambda| < \hat{\lambda},\ |\sigma| < \hat{\sigma},\ |t| < T\right\}$$

for small $\hat{\lambda}$ and $\hat{\sigma}$, because $\sigma_0(\lambda)$ is holomorphic in a neighborhood of $\lambda = 0$ and $\sigma_0(0) = 0$. For the derivative we obtain

$$\frac{\partial S(0,0,t)}{\partial \sigma} = 1 - 2(v_1 - v_2)^2\sigma_0'(\lambda_0^0)\exp(\frac{1}{2}kt(v_1 - v_2))$$
$$\cdot(1 - \exp(-\frac{1}{2}kt(v_1 - v_2)))[v_1\exp(-\frac{1}{2}kt(v_1 - v_2)) - v_2]^{-1}$$
$$\cdot\left[(\sigma_0(\lambda_0^0) - v_2)(1 - \exp(\frac{1}{2}kt(v_1 - v_2))) + (v_1 - v_2)\exp(\frac{1}{2}kt(v_1 - v_2))\right]^{-2} \quad (2.63)$$

where

$$\lambda_0^0 = \lambda_0|_{\lambda=0,\sigma=0} = kv_1t + 2\log\left(\frac{v_1}{v_1 - v_2}\left(\exp(-\frac{1}{2}kt(v_1 - v_2)) - 1\right) + 1\right)$$

and $0 \le |t| \le T$. By analysing this expression for $\frac{\partial S(0,0,t)}{\partial \sigma}$ with the conditions of the lemma taken into account, we conlude that

$$\frac{\partial S(0,0,t)}{\partial \sigma} \neq 0$$

for all $|t| \le T$. This last assertion is especially descriptive when $\varepsilon = 0$: in this case we have $v_2 = 0$ and $v_1 = 2G$. Then

$$\frac{\partial S(0,0,t)}{\partial \sigma} = 1 - \sigma_0'(0)G^{-1}(1 - \exp(-Gkt)) \neq 0$$

if all the conditions of Lemma 2.4 hold.
Using the implicit function theorem, we establish the existence of a solution to (2.59),(2.60) which is unique and analytic in the polycircle

$$\left\{(\lambda, t):\ |\lambda| < \hat{\lambda},\ |t| < T\right\}$$

for $\hat{\lambda}$ small enough.
If $\delta(\lambda) \neq 0$ then we can easily show (similar to obtaining the inequality (2.52)) that $\sigma < \tilde{\sigma}$ where

$$\tilde{\sigma}_t = k\left(\frac{1}{2}\tilde{\sigma}^2 + \tilde{\sigma}\tilde{\sigma}_\lambda - G_1\tilde{\sigma}\right) + \varepsilon,$$

$$\tilde{\sigma}_0(\lambda) > \sigma_0(\lambda),\ \lambda > 0,\ \tilde{\sigma}_0(0) = 0$$

with

$$G_1 = G - \sup_{0 \le \lambda \le \hat{\lambda}} \delta(\lambda).$$

Then, by repeating the above arguments, Lemma 2.4 can similarly be proved.

Applying this Lemma to the problem (2.50),(2.51) with $\varepsilon = 0$, $\delta = 0$, $G = N_1$, we obtain that for all $t, \in [0, T]$ and $\lambda \in [0, \hat{\lambda})$:

$$\sigma(\lambda, t) \le \text{const.} \tag{2.64}$$

From (2.52), (2.64) we establish the correlation

$$\int_0^\infty (\exp(\lambda x) - 1)c_n(x,t)dx \le \text{const}, 0 \le \lambda < \hat{\lambda},\ 0 \le t \le T,\ n \ge 1. \quad (2.65)$$

Consequently, (2.47) follows from (2.65) and (2.9). Hence, $c \in \Omega^+(T)$:

$$\int_0^\infty \exp(\lambda x)c(x,t)dx \le \text{const}, \quad 0 \le \lambda < \bar{\lambda}, \quad 0 \le t \le T. \quad (2.66)$$

The proof of Theorem 1 is now complete.

Remark 2.1 *It is worth pointing out that the solution does not belong to* $\Omega_\lambda(T)$ *even if* $c_0 \in \Omega_\lambda(0)$. *Actually, for the constant kernels* $K \equiv 1,\ F \equiv 0$ *we obtain from (1.1):*

$$\frac{d\sigma}{dt} = \frac{1}{2}\sigma(t)^2$$

where

$$\sigma(t) = \int_0^\infty (\exp(\lambda x) - 1)c(x,t)dx.$$

Hence, $\sigma(t) \to \infty$ *as* $t \to 2/\sigma(0) < \infty$. *Consequently, the right "tails" of solutions (i.e. for large values of x) increase in time. This growth is fast enough for the solution to leave* $\Omega_\lambda(T)$ *within a finite time but it is sufficiently slow to remain inside* $\Omega(T)$ *for all* $T > 0$.

3 Mass conservation

Theorem 2 *Let the conditions of Theorem 1 hold and suppose that* $r \ge 2$. *If, in addition,*

$$\int_0^x yF(x-y,y)dy \le const \cdot (1 + x^r) \quad (3.1)$$

then the mass conservation law holds.

Proof. We are ready now to improve the inequality (2.40) and demonstrate that for all $t \ge 0$ the function $c(x,t)$ yields, similarly to (2.7), the mass conservation law

$$N_1 = \int_0^\infty xc(x,t)dx = \text{const}.$$

This equality holds due to the boundedness of the upper moments of $c(x,t)$ for all $t \geq 0$ (see (2.39)). Actually, by integrating (1.1) with weight x, we obtain

$$\frac{dN_1(t)}{dt} = -\lim_{n\to\infty}\int_0^n \int_{n-x}^\infty (xK(x,y)c(x,t)c(y,t) - xF(x,y)c(x+y,t))dydx.$$

Passing to the limit we obtain zero if the integrals

$$\int_0^\infty \int_0^\infty xK(x,y)c(x,t)c(y,t)dxdy \quad \text{and} \quad \int_0^\infty \int_0^\infty xF(x,y)c(x+y,t)dxdy$$

are bounded. The first integral with the coagulation kernel is bounded due to (2.1) and boundedness of the second moment N_2. For the integral with the fragmentation kernel we appeal to (2.39) and (3.1) to see that

$$\int_0^\infty \int_0^\infty xF(x,y)c(x+y,t)dydx = \int_0^\infty c(x,t)\int_0^x yF(x-y,y)dydx \leq \text{const}(\bar{N}_0+\bar{N}_r).$$

This proves Theorem 2.

Remark 3.1 *If at a critical time $t_c < \infty$ the second moment $N_2(t)$ had become infinite then the formal integration of (1.1) over $[0,\infty)$ with weight x would give us (in the coagulation part) the indeterminance* $\infty - \infty$ *which would lead to the infringement of the mass conservation law.*

Remark 3.2 *In the well-known example of non-uniqueness and non-conservation of mass [20, 24] with $K \equiv 0,\ F \equiv 2,\ c_0(x) = (\lambda + x)^{-3}, \lambda > 0$ there are two solutions*

$$c(x,t) = \frac{\exp(\lambda t)}{(\lambda + x)^3}, \quad c(x,t) = \exp(-tx)\left(c_0(x) + \int_x^\infty c_0(y)[2t + t^2(y-x)]dy\right)$$

where the initial data does not satisfy the conditions of Theorem 2 since $r < 2$ for such c_0. Therefore the condition on r in Theorem 2 is optimal.

Remark 3.3 *When $K \equiv 0$ and $F(x,y) = 2(x+y)$ then*

$$c(x,t) = \exp(t)\cdot(1+x^2)^{-2}$$

is a solution to (1.1) which is clearly not mass conserving. For this example we can see that from (3.1) we have $r = 3$, but the third moment of the initial distribution is unbounded. This solution also demonstrates that the hypotheses of Theorem 2 are actually the best possible conditions on the fragmentation kernel for the mass conservation law to hold. Other examples of solutions which are not mass conserving can be found in Stewart [21].

4 Uniqueness theorem

Theorem 3 *Let case 2^0 of Theorem 1 hold and suppose $m_1 \leq 1$. Then the solution to the initial value problem (1.1), (1.2) is unique in the class $\Omega(T)$.*

To prove uniqueness we use the following lemma (similar to [9]).

Lemma 4.1 *Let $v(\lambda, t)$ be a real continuous function having continuous partial derivatives v_λ and $v_{\lambda\lambda}$ on $D = \{0 \leq \lambda \leq \lambda_0,\ 0 \leq t \leq T\}$. Assume that $\alpha(\lambda), \beta(\lambda, t), \gamma(\lambda, t)$ and $\theta(\lambda, t)$ are real and continuous on D, having continuous partial derivatives there in λ and that the functions $v, v_\lambda, \beta, \gamma$ are non-negative. Suppose that the following inequalities hold on D:*

$$v(\lambda, t) \leq \alpha(\lambda) + \int_0^t (\beta(\lambda, s) v_\lambda(\lambda, s) + \gamma(\lambda, s) v(\lambda, s) + \theta(\lambda, s)) ds, \quad (4.1)$$

$$v_\lambda(\lambda, t) \leq \alpha_\lambda(\lambda) + \int_0^t \frac{\partial}{\partial \lambda} (\beta(\lambda, s) v_\lambda(\lambda, s) + \gamma(\lambda, s) v(\lambda, s) + \theta(\lambda, s)) ds. \quad (4.2)$$

Let $C_0 = \sup_{0 \leq \lambda \leq \lambda_0} \alpha, C_1 = \sup_D \beta,\ C_2 = \sup_D \gamma,\ C_3 = \sup_D \theta$. Then

$$v(\lambda, t) \leq C_0 \exp(C_2 t) + (C_3/C_2)(\exp(C_2 t) - 1)$$

in any region $R \subset D$:

$$R = \{(\lambda, t) : 0 \leq t \leq t' < T';\ \lambda_1 - C_1 t \leq \lambda \leq \lambda_0 - C_1 t, \quad 0 < \lambda_1 < \lambda_0\}.$$

where $T' = \min\{\lambda_1/C_1), T\}$.

Proof. Let us denote the right-hand side of the inequality (4.1) by $w(\lambda, t)$. By differentiating in t, λ, we obtain from (4.1),(4.2):

$$w_t \leq \beta w_\lambda + \gamma w + \theta \leq c_1 w_\lambda + \gamma w + \theta.$$

Hence for the derivative along the characteristic $\frac{d\lambda}{dt} = -C_1$ we have

$$\frac{dw}{dt} \leq \gamma w + \theta. \quad (4.3)$$

Let us denote $u(t) = \overline{C}_0 \exp(C_2 t) + (\overline{C}_3/c_2)(\exp(c_2 t) - 1)$ with $\overline{C}_0 > C_0$, $\overline{C}_3 > C_3$. Obviously, $u(0) > w(\lambda, 0)$ for all $\lambda \in [0, \lambda_0]$. Let $(\hat{\lambda}, \hat{t})$ be the first point on a characteristic straight line where $w = u$. Then at the point $(\hat{\lambda}, \hat{t})$

$$\frac{d(u - w)}{dt} \leq 0$$

and consequently

$$w_t - C_1 w_\lambda \geq u_t. \tag{4.4}$$

From $u_t = C_2 u + \overline{C}_3$ we can easily see that at the point $(\hat{\lambda}, \hat{t})$ the equality $u_t = C_2 w + \overline{C}_3$ holds. Recalling (4.4), we obtain a contradiction with (4.3):

$$\frac{dw}{dt} = w_t - C_1 w_\lambda \geq C_2 w + \overline{C}_3 > C_2 w + C_3 \geq \gamma w + \theta.$$

This proves Lemma 4.1.

We shall prove the uniqueness of a solution $c \in \Omega^+(T)$ in $\Omega(T)$ by contradiction. Suppose that there are two distinct solutions c and g of the initial value problem (1.1), (1.2) in $\Omega(T)$. Using the notation $u = |c - g|$, $\psi = |c + g|$ and conditions (2.1), we find that

$$\begin{aligned} u(x,t) \leq \int_0^t \Big\{ & \frac{1}{2} k(1+x) \int_0^x u(x-y,s)\psi(y,s)dy + \\ & + \frac{1}{2} k u(x,s) \int_0^\infty (1+x+y)\psi(y,s)dy + \frac{1}{2} k\psi(x,s) \int_0^\infty (1+x+y) u(y,s)dy \\ & + \int_x^\infty F(y-x,x)u(y,s)dy + \frac{1}{2} u(x,s) \int_0^x F(x-y,y)dy \ \Big\} ds. \end{aligned} \tag{4.5}$$

Since $c, g \in \Omega(T)$, we have $u, \psi \in \Omega(T)$, and $u, \psi \geq 0$ on Π. Let $\hat{\lambda} > 0$ be chosen such that

$$\begin{aligned} \int_0^\infty \exp(\hat{\lambda} x) u(x,t)dx &\leq \text{const} < \infty, \\ \int_0^\infty \exp(\hat{\lambda} x) \psi(x,t)dx &\leq \text{const} < \infty \end{aligned} \tag{4.6}$$

uniformly with respect to t, $0 \leq t \leq T$, and let

$$0 \leq \lambda < \hat{\lambda}. \tag{4.7}$$

Integration of inequality (4.5) with the weight $\exp(\lambda x)$ yields

$$\begin{aligned} \int_0^\infty \exp(\lambda x) u(x,t)dx \leq \int_0^t \Big\{ \Big\{ \int_0^\infty \int_0^\infty \frac{1}{2} k(\exp(\lambda(x+y)) + \exp(\lambda x) + \exp(\lambda y)) \\ \cdot (1+x+y) u(x,s)\psi(y,s)dxdy + \int_0^\infty \exp(\lambda x) u(x,s) \cdot \\ \cdot \left(\int_0^x \exp(\lambda y - \lambda x) F(x-y,y)dy + \frac{1}{2} \int_0^x F(x-y,y)dy \right) dx \Big\} ds. \end{aligned}$$

Here we have changed the order of integration in the integral, using Fubini's theorem [7]. We strengthen this inequality with (2.2) with $m_1 \le 1$ taken into account:

$$\int_0^\infty \exp(\lambda x)u(x,t)dx \le \frac{3}{2}\int_0^t \left\{\int_0^\infty \int_0^\infty k\exp(\lambda x+\lambda y)\right.$$
$$\left.\cdot(1+x+y)u(x,s)\psi(y,s)dxdy + b\int_0^\infty (1+x)\exp(\lambda x)u(x,s)dx\right\}ds. \quad (4.8)$$

The following inequality can similarly be proved:

$$\int_0^\infty x\exp(\lambda x)u(x,t)dx \le \frac{3}{2}\int_0^t \left\{\int_0^\infty \int_0^\infty k\exp(\lambda x+\lambda y)(x+y)\right.$$
$$\left.\cdot(1+x+y)u(x,s)\psi(y,s)dxdy + b\int_0^\infty x(1+x)\exp(\lambda x)u(x,s)dx\right\}ds. \quad (4.9)$$

Let

$$U(\lambda,t) = \int_0^\infty \exp(\lambda x)u(x,t)dx; \quad \Psi(\lambda,t) = \int_0^\infty \exp(\lambda x)\psi(x,t)dx.$$

The functions U and Ψ are analytic in the half-plane $Re(\lambda) < \hat{\lambda}$ for any fixed t, $0 \le t \le T$. Let λ be on the real axis and satisfy

$$0 \le \lambda \le \lambda_0 < \hat{\lambda}. \quad (4.10)$$

The inequalities (4.6) then ensure that, for any integer $i \ge 1$,

$$\sup_{0\le t\le T, 0\le\lambda\le\lambda_0} \left\{\frac{\partial^i}{\partial\lambda^i}U(\lambda,t),\ \frac{\partial^i}{\partial\lambda^i}\Psi(\lambda,t)\right\} < \infty. \quad (4.11)$$

Moreover, since $u(x,t)$ and $\psi(x,t)$ are continuous on Π and inequalities (4.6) are satisfied, for a given $\varepsilon > 0$ there are corresponding numbers $\delta(\varepsilon) > 0$ and $\delta_i(\varepsilon) > 0$ such that, if $0 \le t, t' \le T$, and $i \ge 1$, then

$$\sup_{0\le\lambda\le\lambda_0} \{|U(\lambda,t') - U(\lambda,t)|, |\Psi(\lambda,t') - \Psi(\lambda,t)|\} < \varepsilon, \quad |t'-t| < \delta,$$
$$\sup_{0\le\lambda\le\lambda_0} \left\{\left|\frac{\partial^i}{\partial\lambda^i}U(\lambda,t') - \frac{\partial^i}{\partial\lambda^i}U(\lambda,t)\right|, \left|\frac{\partial^i}{\partial\lambda^i}\Psi(\lambda,t') - \frac{\partial^i}{\partial\lambda^i}\Psi(\lambda,t)\right|\right\} < \varepsilon, \quad (4.12)$$
$$|t'-t| < \delta_i.$$

In fact, to show (4.12) it is enough to split the integrals in (4.6) and use the uniform smallness of the "tails" $\int_z^\infty$, which holds due to (4.7), (4.10) and the inequality (2.25) with, for example, $\psi(x) = \exp(\frac{1}{2}(\hat{\lambda} - \lambda_0)x)$. It follows from (4.11),(4.12) that U and Ψ are continuous together with all their partial derivatives with respect to λ in $D = \{0 \leq \lambda \leq \lambda_0,\ 0 \leq t \leq T\}$. Inequalities (4.8),(4.9) imply that

$$U(\lambda,t) \leq \frac{3}{2}\int_0^t \{(k\Psi(\lambda,s)+b)U_\lambda(\lambda,s)+(k\Psi(\lambda,s)+k\Psi_\lambda(\lambda,s)+b)U(\lambda,s)\}ds,$$

$$U_\lambda(\lambda,t) \leq \frac{3}{2}\int_0^t \frac{\partial}{\partial\lambda}\{(k\Psi + b)U_\lambda + (k\Psi + k\Psi_\lambda + b)U(\lambda,s)\}ds,$$

and U and Ψ are non-negative in D together with their partial derivatives with respect to λ. We can thus apply Lemma 4.1 in D. Let

$$c_1 = \frac{3}{2}(k\sup_D \Psi + b),\ c_2 = \frac{3}{2}k\sup_D(\Psi + \Psi_\lambda) + \frac{3}{2}b.$$

Then $U(\lambda,t) = 0$ in the region R defined in Lemma 4.1. Since $u(x,t)$ is continuous, $u(x,t) = 0$ for $0 \leq t \leq t',\ 0 \leq x < \infty$; hence $U(\lambda,t) = 0$ not only in R, but for all $0 \leq \lambda \leq \lambda_0,\ 0 \leq t \leq t'$. Applying the same reasoning to the interval $[t', 2t']$, we conclude that $u(x,t) = 0$ for $0 \leq t \leq 2t',\ 0 \leq x < \infty$ and, continuing this process, we establish that $u(x,t) = 0$ on Π, that is, $c = g$ on Π. This completes the proof of Theorem 3.

Remark 4.1 *In the example of non-uniqueness from Remark 3.2 the initial data does not satisfy the conditions of Theorem 3.*

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